ETH Zürich
Institute of Theoretical Computer Science
Dr. J. Lengler
H. Einarsson, R. Nenadov

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Exercise sheet 0

Graphs and Algorithms

Exercise 1 (Connectivity)

Let G be a graph. Prove the following statements.

(a) Let s, t be two different vertices of G. If there is a walk starting in s and ending in t then there is a path with leaves s and t.

(b) We define

$$s \sim t \quad :\Leftrightarrow \quad \text{there exists a path from } s \text{ to } t.$$

Then "$\sim$" is an equivalence relation, i.e., it is reflexive, symmetric and transitive.

(c) G is the disjoint union of its connected components.

Exercise 2 (Properties of Trees)

Prove the following statements:

(a) Let T be a tree.

  (i) If $n := |V(T)| \geq 2$ then T contains at least two leaves.

  (ii) Deleting a leaf from T produces another tree.

(b) *(Characterization of trees)* For a graph G on n vertices, the following are equivalent:

  - G is connected and has no cycles.
  - G is connected and has $n-1$ edges.
  - G has $n-1$ edges and no cycles.
  - For each $u, v \in V(G)$, G has exactly one u, v-path.

(c) Every edge of a tree is a bridge.

(d) Adding one edge (and no vertices) to a tree forms exactly one cycle.

(e) Every connected graph contains a spanning tree.

Exercise 3 (Bridge-It)

(a) In a Bridge-it game, show that when no more moves are possible then exactly one player has built a bridge.

(b) Describe an explicit winning strategy for player 1. (This includes proving that your strategy is successful.)

Exercise 4 (Strategy Stealing)

In the lecture it was shown by a strategy stealing argument that the player who makes the first move wins Bridge-it.

Claim. *The player who makes the second move wins Bridge-it.*

Proof (strategy stealing). Assume for the sake of contradiction that Player 1 has a winning strategy. After Player 1 made his first move, Player 2 ignores this move and pretends to be Player 1 by stealing his winning strategy. Hence, Player 2 wins the game, which contradicts our assumption. □

Where is the mistake in this proof?

Exercise 5 (Bridg-it on Graphs)

Consider the following game on a graph G. There are two players, a red color player R and a blue color player B. Initially all edges of G are uncolored. The two players alternately color an uncolored edge of G with their color until all edges are colored. The goal of B is that in the end, the blue-colored edges form a connected spanning subgraph of G. The goal of R is to prevent B from achieving his goal. Assume that R starts the game.

Note that it was essentially shown in the lecture that B can always win if G contains two edge-disjoint spanning trees. (Recall the winning strategy for Player 1 in Bridg-it.)

Prove that on the other hand, R can always win if G does not contain two edge-disjoint spanning trees!

Discussion of the solution in the exercise class on 21.2.2013.