Graphs and Algorithms

Exercise 1 (Tree embedding)

Given a graph G with minimum degree $\delta(G) \ge d$, show that it contains every tree on d+1 vertices as a subgraph.

Exercise 2 (Bridges)

Prove that a graph G on n vertices can have at most n-1 bridges.

Exercise 3 (Maximal graphs)

For any $n \in \mathbb{N}$, we define the family of graphs \mathcal{G}_n as follows,

 $\mathcal{G}_n = \{G \mid G \text{ is a graph on } 2n \text{ vertices, with exactly } n \text{ vertices of degree } 1\}.$

We say that a graph $G \in \mathcal{G}_n$ is *maximal* if there is no graph $G' \in \mathcal{G}_n$ such that G is a proper subgraph of G'. Prove or disprove the following statement: for every $n \in \mathbb{N}$, n > 0, every maximal graph in \mathcal{G}_n has exactly $\binom{n+1}{2}$ edges.

Hint: First read the following proof. Is it correct?

Proof. We prove that statement by induction on n.

For n = 1, \mathcal{G}_1 is empty since graphs on 2 vertices have either 0 or 2 vertices of degree 1.

For n = 2, there are exactly three graphs in \mathcal{G}_2 , namely the paths of length 3, 2, and 1, together with 0, 1, and 2 isolated vertices (vertices of degree 0), respectively. The only maximal graph is the path of length 3, so the claim holds.

Assume now that it holds for some n-1, $n \geq 3$. In order to construct a maximal graph $G_n \in \mathcal{G}_n$, we can start with a maximal graph $G_{n-1} \in \mathcal{G}_{n-1}$. We need to add two new vertices to G_{n-1} , one vertex u of degree 1 and one vertex v of degree $\neq 1$. If u and v are adjacent, then u is not connected to any further vertices. In order for G_n to be maximal, v needs to be connected to all vertices of degree > 1. There can be no vertex of degree 0, so v is connected to n-1 further vertices. Therefore, $e(G_n) = \frac{n(n-1)}{2} + 1 + (n-1) = \frac{n^2+n}{2} = \binom{n+1}{2}$.

If u and v are not adjacent, then u must be connected to a vertex of degree > 1, and as before, v must be connected to all n-1 vertices of degree > 1. Thus, we have again $e(G_n) = \frac{n(n-1)}{2} + 1 + (n-1) = \binom{n+1}{2}$, and the claim is proven. \Box

Exercise 4 (Cops and robber)

There is a rumor that the greatest thief of all times will arrive in Zürich this morning, and as a chief of a local Police Department, you want to finally catch him and get a big promotion. To simplify the scenario, assume that the map of the city is represented as a graph G, where edges are streets and each vertex represents a building (bank, watch shop, Sprüngli, ...). Before he arrives, you can choose any number of cops from your station and place them to guard some buildings.

After a while, the great robber has finally arrived to his apartment in some building v. Being very clever, he learned the positions of all your cops and chose v accordingly. And now the chase starts ... Assuming that both cops and robber know each other positions at every step, first each cop may move to an adjacent building (or stay put) and then the robber. If at any point a cop and the robber happen to be in the same building, the robber is caught.

Note that more than one cop can be in the same building at any time. Further, moves are not simultaneous, i.e. the robber has first to wait until all cops make their move.

Prove that if girth of graph G is at least 5, you require at least $\delta(G)$ many cops to catch the robber, where $\delta(G)$ denotes the minimum degree of vertices in G.

Exercise 5 (Spanning trees (D. West, 2.2.6))

Let G be the 3-regular graph with 4m vertices formed from m pairwise disjoint kites by adding m edges to link them in a ring, as shown in the figure for m = 6. Prove that the number of spanning trees of G is $2m \cdot 8^m$.



DISCUSSION OF THE SOLUTION IN THE EXERCISE CLASS ON 28.2.2013.