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Exercise sheet 2

Graphs and Algorithms

Exercise 1 (Large bipartite subgraph)

Let G be a graph on n vertices with m edges. Show that there exists a partition of the vertices into two disjoint sets A and B such that

$$|\{\{u,v\} \in E \mid u \in A, v \in B\}| \geq \frac{m}{2}.$$

Exercise 2 (Graph isomorphisms (Bondy, Murty, 1.2.15))

Let G and H be isomorphic graphs on n vertices, let θ be an isomorphism between G and H, and let α be an automorphism on G.

(a) Show that $\theta\alpha$ is an isomorphism between G and H.

(b) Deduce that the set of all isomorphisms between G and H is the coset $\theta\mathrm{Aut}(G) = \{\theta\alpha \mid \alpha \in \mathrm{Aut}(G)\}$ of $\mathrm{Aut}(G)$.

(c) Deduce that the number of labelled graphs isomorphic to G is equal to $\frac{n!}{\#\mathrm{Aut}(G)}$.

(d) Let $g(n)$ be the number of isomorphism classes of graphs on n vertices. Deduce from c that
$$\frac{2^{\binom{n}{2}}}{n!} \leq g(n) \leq 2^{\binom{n}{2}}.$$
Conclude that $\log(g(n))$ is asymptotically equal to $\frac{\log 2}{2}n^2$.

Note: Two real-valued functions f and g are *asymptotically equal* if $f(n) = (1 + o(1))g(n)$ (as $n \to \infty$).

Exercise 3 (Tree Counting)

(a) Given positive integers $d_1, \ldots, d_n$ summing to $2n-2$, how many trees with vertex set $\{1, \ldots, n\}$ are there such that vertex i has degree d_i for each i?

(b) How many labelled trees on n vertices with exactly 3 leaves are there?

(c) How many labelled trees on the vertex set $\{1, \ldots, n\}$ are there which contain the edge $\{3, 4\}$?

(d) How many spanning trees does the labelled $K_{n,m}$ have?

Exercise 4 (Block graphs are trees)

Let $G = (V, E)$ be a connected graph. Let E' be a maximal set of edges of G such that any two edges lie on a common cycle. The graph induced by E' is called a *block* of G.

The *block graph* of G is a bipartite graph with vertex set $A \cup B$ where A is the set of articulation points and B the set of blocks of G. There is an edge between $a \in A$ and $b \in B$ if and only if $a \in V(B)$.

The goal of this exercise is to prove that the block graph of every connected graph is a tree. Prove the following claims and from them deduce the statement.

(a) The intersection of two blocks consists of at most one vertex and this vertex is an articulation point.

(b) Each edge is contained in exactly one block.

(c) The block graph is connected.

(d) The block graph is acyclic.

Exercise 5 (Cops and robber 2)

As you proved last week, if the city map (for our purposes a graph G) has girth at least 5, then at least $\delta(G)$ cops are needed in order to catch the robber. However, it turned out that the robber doesn't know this fact, and thus has selected a city without this property (city maps are given below). It is up to you now again to choose the minimal number of cops in order to guarantee that the robber will be caught (eventually)!

What is the minimal number of cops such that you can catch the robber if

(a) the graph G is an $n \times n$ grid,

(b) the graph G is an $n \times n \times n$ grid.

Note: Vertices of G are elements of the set $\{1, \ldots, n\}^k$, for $k = 2$ (a) and $k = 3$ (b), with an edge between two elements if and only if they differ in exactly one coordinate by exactly 1. For example, there is an edge between $(1, 1, 2)$ and $(1, 2, 2)$, but not between $(1, 1, 2)$ and $(1, 2, 1)$.

Proving that you can catch the robber with c cops is not enough! You also have to prove that he can always escape if you use $c - 1$ cops instead!

DISCUSSION OF THE SOLUTION IN THE EXERCISE CLASS ON 7.3.2013.