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Exercise sheet 4

Graphs and Algorithms

Exercise 1 (Vertex vs. edge connectivity)

For all $t \in \mathbb{N}$ show that there exists a graph with vertex connectivity 1 and edge connectivity t.

Exercise 2 (Exam question, 2010.)

Let $k \geq 1$ and let G be a graph on n vertices such that any two distinct vertices x and y satisfy $\deg(x) + \deg(y) \geq n + k - 2$. Prove that G is k-connected.

Exercise 3 (The k-dimensional grid)

Is the k-dimensional grid (as defined in exercise 5 of sheet 2), k-connected?

Exercise 4 (Dilworth's theorem)

Let $(P, \leq)$, where P is a finite set, be a partially ordered set. To recapitulate, $(P, \leq)$ is a partially ordered set (or *poset*) if the relation $\leq$ on P satisfies

1. $a \leq a$,
2. $a \leq b$ and $b \leq a$ implies $a = b$ and
3. $a \leq b$ and $b \leq c$ implies $a \leq c$.

Note that we do *not* require that for all pairs (a, b) one of $a \leq b$ and $b \leq a$ holds, as we would do for a total ordering.

We say that a subset $C \subseteq P$ is a *chain* if $a \leq b$ or $b \leq a$ for every two elements $a, b \in C$. Similarly, we call a subset $A \subseteq P$ an *antichain* if neither $a \leq b$ nor $b \leq a$ for every two distinct elements $a, b \in A$. Finally, we say that a set $C_1, \ldots, C_t$ of chains covers P if every element of P belongs to some chain C_i.

Prove that the maximum number of elements in an antichain equals to the minimum number of chains needed to cover P.

Hint 1: Prove that we need at least as many chains as there are elements in an antichain. (This is easy.)

Hint 2: Consider the auxiliary bipartite graph $G = (A \cup B, E)$, where A and B are disjoint copies of P, with an edge between $a \in A$ and $b \in B$ if $a \leq b$ and a, b represent distinct elements in P. Construct an antichain in $(P, \leq)$ from each vertex cover in G. What about matchings in G and sets of covering chains?

Discussion of the solution in the exercise class on 21.03.2013.