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FS 2013
Graded homework 1

# Graphs and Algorithms

## Exercise 1 (Combining $k$-connected graphs)

Let $G_1$ and $G_2$ be $k$-connected graphs and let $K_k$ be the complete graph on $k$ vertices. Now consider the graph with vertex set $V(G_1) \cup V(K_k) \cup V(G_2)$ where we connect each vertex of $K_k$ to one vertex in $G_1$ and one vertex in $G_2$ such that no two vertices in $K_k$ are neighbors of the same vertex in $G_1$ or $G_2$. Is the resulting graph always $k$-connected?

## Exercise 2 (Characterizations of forests)

Let $G = (V, E)$ be a graph. Prove that the following statements are equivalent.

(a) $G$ is a forest. (In other words, $G$ is acyclic.)

(b) Every connected subgraph is an induced subgraph.

(c) Every induced subgraph has a leaf.

(d) The number of connected components is equal to $|V| - |E|$.

## Exercise 3 (No small cycles, few edges)

Goal of this exercise is to show that a graph with "large" girth cannot be too dense. More formally, let $G = (V, E)$ be a connected graph on $n$ vertices and $m$ edges with girth at least $2(k + 1)$, and consider a subset $V' \subseteq V$ such that $\delta(G[V']) \geq \rho$, where $\rho = m/n$ (we know that such subgraph exists from the exercise sheet 3).

(a) For $v \in V'$ consider the set

$$V'' = \{w \in V' \mid d(v, w) \leq k\},$$

where $d(v, w)$ is the length of a shortest path starting in $v$ and ending in $w$. Prove that $G[V'']$ is a tree.

(b) Assuming $\rho > 2$, prove that $|V''| \geq 1 + \rho \cdot \frac{(\rho-1)^k - 1}{\rho - 2}$.

(c) Deduce from (a) and (b) that $\frac{m}{n} \leq n^{\frac{1}{k}} + 1$. Note that this gives an upper bound $m \leq n^{1+1/k} + n$.

## Exercise 4 (Larger bipartite subgraph)

Let $G = (V, E)$ be a connected graph on $n$ vertices and $m$ edges, and let $b(G)$ denote the maximal number of edges in a bipartite subgraph of $G$,

$$b(G) = \max_{\substack{A,B\subseteq V \\ A\cap B=\varnothing}} \left|\{e = \{u, v\} \in E \mid u \in A, v \in B\}\right|.$$

In one of the previous exercise sheets we showed that $b(G) \geq m/2$. Now we want to prove a stronger claim,

$$b(G) \geq \frac{m}{2} + \frac{n-1}{4}.$$

*Hint*: use induction on the number of vertices of $G$, and distinguish the following cases.

(a) $G$ has an articulation vertex.

(b) $G$ has a vertex which is not an articulation vertex and has an odd degree. In this case, prove that even $b(G) \geq \frac{m}{2} + \frac{n}{4}$.

(c) $G$ has no articulation vertex and no vertex of odd degree.

Note that this graded homework sheet counts 10% towards your final grade. As such this is considered as part of your exam and any attempt to cheat will be dealt with under ETH exam regulations. While you are allowed to talk to fellow students and consult books or the Internet, we expect you to hand in your own independent write-up.

Hand in your solutions as a LaTeX-typeset .pdf file (also attach your .tex file) by mail at `heinarss@inf.ethz.ch` or `rnenadov@student.ethz.ch` until Wednesday 27.3.2013, before 23:59.