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Proposed solutions for graded HW 1

Graphs and Algorithms

Exercise 1 (Combining k-connected graphs)

Let $G = (V, E)$ be the combined graph, as given in the exercise, and let $S \subseteq V$ be an arbitrary subset of at most $k-1$ vertices (the set of *removed* vertices). Since G_1, G_2 and K_k are k-connected, then $G_1[V \setminus S]$, $G_2[V \setminus S]$ and $K_k[V \setminus S]$ are connected. Thus it remains to show that for any two vertices $a \in V(G_1) \setminus S$ and $b \in V(G_2) \setminus S$ there exists a path which avoids S (we will implicitly prove that this also holds if either a or b are in $V(K_k) \setminus S$).

Since we removed at most $k-1$ vertices and there are k edges connecting G_1 and K_k we know that there is at least one edge $e = \{a', k'\}$ where $a' \in V(G_1) \setminus S$ and $k' \in V(K_k) \setminus S$. Now since $G_1 \setminus S$ is still connected there exists a path P' from a to a'. By a symmetric argument there exists an edge $e' = \{k'', b''\}$, with $k'' \in K_k \setminus S$ and $b'' \in G_2 \setminus S$, and a path P'' which connects b to b''. Since $K_k \setminus S$ is connected we have that there is a path P''' connecting k' and k''. Concatenating P', P''' and P'' now yields a path from a to b proving that the graph is still connected.

Exercise 2 (Characterizations of forests)

(a)⇒(d): For ease of notation let's denote the number of components of G by $c(G)$. We proceed by induction on $|E(G)|$. For the base case when $|E(G)| = 0$ every vertex is isolated and we have $|V|$ components which equals $|V(G)| - |E(G)|$.

Now let's assume that $|E(G)| > 0$ and let $e \in E(G)$ be an arbitrary edge. Since e is a bridge, $G - e$ has one more component than G so the induction hypothesis now yields that

$$\begin{aligned} c(G) &= c(G-e) - 1 \\ &= |V(G-e)| - |E(G-e)| - 1 \\ &= |V(G)| - |E(G)| \end{aligned}$$

which proves that the induction hypothesis holds.

(b)⇒(a): Let's assume that every connected subgraph of G is induced. For the sake of contradiction let's assume that G has a cycle with vertices from the set $S = \{v_0, v_1, \ldots, v_t\}$ such that $\{v_i, v_{i+1}\} \in E$, for $i \in \{0, 1 \ldots, t-1\}$, and $\{s_0, s_t\} \in E$. Now $G[S]$ contains a cycle, but $G[S] - e$, where e is an edge on the cycle, is connected but not induced which contradicts the assumption, thus G is a forest.

(c)⇒(b): Let's assume that every induced subgraph has a leaf. Again for the sake of contradiction let's assume there exists a connected subgraph H of G which is not induced. This in turn implies that $G[V(H)]$ contains a cycle because if G were a forest then every induced subgraph on G would also be a forest and hence every connected subgraph of G would be induced (proving that (a) implies (b)). Now let C be the vertices on a cycle in $G[V(H)]$, then $G[C]$ contains a cycle and each vertex in $G[C]$ has degree at least two contradicting the assumption.

(d)⇒(c): Let's assume that the number of components is equal to $|V| - |E|$. For the sake of contradiction let's assume that there exists an induced subgraph $G[S]$ which does not have a leaf. This in turn implies that there exists an edge $e \in G[S]$ which is not a bridge. Let's consider $G \setminus e$. If it still contains a non-bridge edge we remove it and iterate until what remains is a graph where each edge is a bridge. Let E' be the set of edges we removed, and note that $E' \neq \varnothing$. Then $G \setminus E'$ is a forest since each edge is a bridge and the number of components is $|V| - (|E| - |E'|) > |V| - |E|$, contradicting the assumption above.

Exercise 3 (No small cycles, few edges)

(a) By definition $G[V'']$ is connected. If $G[V'']$ contained a cycle the cycle would have length at most $2k+1$ contradicting the fact that the girth of G is at least $2(k+1)$. To see this consider the spanning tree obtained by doing a BFS on $G[V'']$. If any edge $\{u,v\}$ is added to this tree then the unique paths to u and v in the spanning tree along with the edge will create a cycle of length at most $2k+1$ contradicting the girth condition. Hence since $G[V'']$ is connected it is a tree.

(b) Now let's consider the tree $G[V'']$ obtained in part (b). We will now give a lower bound on the number of vertices in the $G[V'']$. Note that since the minimum degree of $G[V'']$ is at least ρ the root has at least ρ children and each child has at least $\rho-1$ children. Thus in the tree we have for $\rho > 2$ at least

$$1+\rho+\rho(\rho-1)+\cdots+\rho(\rho-1)^{k-1}=1+\rho\left(\frac{(\rho-1)^k-1}{\rho-2}\right)$$

vertices (the sum is geometric which yields the formula).

(c) Now if $\rho \leq 2$ the bound trivially holds so we can assume that $\rho > 2$. We now have using the result from part (b) that

$$\begin{aligned}
& n \geq 1+\frac{m}{n}\left(\frac{\left(\frac{m}{n}-1\right)^k-1}{\frac{m}{n}-2}\right) \\
\Rightarrow\ & m-2n \geq \frac{m}{n}\left(\frac{m}{n}-1\right)^k-2 \\
\Rightarrow\ & n-2n\cdot\frac{n-1}{m} \geq \left(\frac{m}{n}-1\right)^k \\
\Rightarrow\ & n \geq \left(\frac{m}{n}-1\right)^k \\
\Rightarrow\ & n^{\frac{1}{k}}+1 \geq \frac{m}{n}.
\end{aligned}$$

Which is the desired result.

Exercise 4 (Larger bipartite subgraph)

Let $G=(V,E)$ be a connected graph on n vertices and m edges. As suggested in the exercise, we will prove by induction on n that $b(G) \geq \frac{m}{2}+\frac{n-1}{4}$. For the base case $n=1$ the claim trivially holds, and let us assume that it holds for all $n' < n$.

Proof of part (a). Let $v \in V$ be an articulation point of G, and let $C_1,\ldots,C_t \subseteq V$, where $t \geq 2$, be vertices of connected components of $G-v$. Since $|C_i| \geq 1$ for every $i \in [t]$, we also have $|C_i \cup v| \leq n-1$. Thus, by induction hypothesis we have $b(G[C_i \cup v]) \geq \frac{m_i}{2}+\frac{n_i-1}{4}$ for every $i \in [t]$, where n_i and m_i are the number of vertices and edges in $G[C_i \cup v]$, respectively. Simple calculation now yields that

$$b(G) \geq \sum_{i=1}^{t} \frac{m_i}{2}+\frac{n_i-1}{4}=\sum_{i=1}^{t}\left(\frac{m_i}{2}+\frac{n_i}{4}\right)-\frac{t}{4}.$$

Further, observe that every edge of G belongs to exactly one induced subgraph $G[C_i \cup v]$, thus $\sum_{i=1}^{t} m_i = m$. Finally,

$$\sum_{i=1}^{t} n_i = \sum_{i=1}^{t} (|C_i| + 1) = n - 1 + t,$$

hence we get

$$b(G) \geq \sum_{i=1}^{t} \left(\frac{m_i}{2} + \frac{n_i}{4} \right) + \frac{t}{4} \geq \frac{m}{2} + \frac{n - 1 + t - t}{4} = \frac{m}{2} + \frac{n-1}{4}.$$

□

Proof of part (b). Let $v \in V$ be a vertex of G such that v has an odd degree and is not articulation point. Since $G - v$ is connected, by induction we have $b(G - v) \geq \frac{m - \deg(v)}{2} + \frac{n-2}{4}$, and further let $A, B \subseteq V \setminus \{v\}$, $A \cap B = \varnothing$, be partitions which prove this. Since $\deg(v)$ is odd, we either have $|\Gamma(v) \cap A| \geq \frac{\deg(v)+1}{2}$ or $|\Gamma(v) \cap B| \geq \frac{\deg(v)+1}{2}$, thus we can either add v to A or B such that the number of additional edges in the bipartite subgraph is at least $\frac{\deg(v)+1}{2}$. Together with the induction hypothesis this yields

$$b(G) \geq b(G - 1) + \frac{\deg(v) + 1}{2} \geq \frac{m - \deg(v)}{2} + \frac{n-2}{4} + \frac{\deg(v)+1}{2} = \frac{m}{2} + \frac{n}{4},$$

which proves the part (b) (*note that we get a slightly stronger result which we need for part (c)*). □

Proof of part (c). If G has a vertex which either satisfies requirement of (a) or (b), we are done. Therefore, we can assume that all vertices in G have an even degree, and further contains no articulation point. Our idea is to find two vertices $u, v \in V$ such that $\{u, v\} \in E$ and further $G[V - \{u, v\}]$ is a connected graph. Let us for the moment assume that we have such two vertices. Then $G[V - \{u\}]$ is a connected graph which contains a vertex v such that v has an odd degree and is not an articulation point. Thus from the part (b) and $|V - \{u\}| = n - 1$ we have

$$b(G - u) \geq \frac{m - \deg(u)}{2} + \frac{n-1}{4}.$$

Similarly as in the previous part, we can add vertex u such that at it "contributes" at least $\frac{\deg(u)}{2}$ additional edges in the bipartite subgraph, and therefore

$$b(G) \geq b(G - u) + \frac{\deg(u)}{2} \geq \frac{m - \deg(u)}{2} + \frac{n-1}{4} + \frac{\deg(u)}{2} = \frac{m}{2} + \frac{n-1}{4}.$$

Thus it only remains to show that we can always find such vertices u and v. Let $u \in V$ be any vertex and consider the block graph of $G - u$. Observe that the trivial case when the whole graph $G - u$ is one block easily implies that u together with any neighbor satisfies the requirement. Thus we can assume that $G - u$ has at least one articulation point, and let b be a block and a an articulation point such that b is a leaf and $\{a, b\}$ is an edge in the block graph. Further, let B be vertices associated with the block b, without the vertex a, and note that $|B| \geq 1$. Since a is an articulation point in $G - u$, and G is the connected graph without an articulation point, there has to exists an edge $\{v, u\} \in E$ such that $v \in B$. Since v is not an articulation point in $G - u$, we conclude that $G[V - \{u, v\}]$ remains connected, thus proving that v, u satisfy requirements. $\square$

Remark: $b(G) \geq \frac{m}{2} + \frac{n-1}{4}$ is known as the Edwards-Erdős bound.