ETH Zürich
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FS 2013
Graded homework 2

# Graphs and Algorithms

## Exercise 1 (Saturating Matching)

Let $G = (A \cup B, E)$ be a bipartite graph with partite sets $A$ and $B$. Assume that $\delta(G) \geq 1$ and $\deg(a) \geq \deg(b)$ whenever $\{a, b\} \in E$ and $a \in A$. Show that $G$ contains a matching that saturates all vertices in $A$.

## Exercise 2 (Brute forcing a lock)

You are presented with a locked pin lock which accepts a four digit pin code. When a digit is pressed the lock evaluates the last four pressed digits and if they match the pass code the lock opens. For example if the pin code is '1234' then pressing 567321234 will open the lock.

Now you would like to open the lock and since you don't know the pin code your only option is to brute force and try all possibilities. But of course, since you are a graph theoretician you would like to press as few keys as possible in the worst case scenario. What is the minimum number of key presses for a 4-digit pin code? It is obvious that at most 40000 key presses will do by typing each possible code in separately but can you do better?

## Exercise 3 (Minimum Degree Game)

Let $G = (V, E)$ be a graph, and consider the following game for two players. In the beginning all edges of $G$ are white, and the players are performing the following – first Player One chooses a white edge and colours it red, then Player Two chooses a white edge and colours it blue, and so on until there is no white edge remaining. In the end every edge is either red or blue, and let us denote the set of red edges with $R$ and the set of blue edges with $B$. We say that the game is a $k$-win for the first player if $\delta(G_r) \geq k$, where $G_r = (V, R)$, and a $k$-win for the second player otherwise.

Prove that if $\delta(G) \geq 4 \cdot k$, for $k \in \mathbb{N}$, then no matter what the second player does, the first player can achieve a $k$-win.

*Hint: Try first to solve the case when every vertex in $G$ has even degree. If there are some vertices with an odd degree, try to reduce it to the previous case.*

## Exercise 4 (Message propagation using edge ID's)

Let $G = (V, E)$ be a graph. Now let $\gamma : E \to \mathbb{N}$ be an injective map (that is, each edge has a unique identifier). Consider the following synchronous[1] message propagation algorithm in the graph:

**Algorithm 1** Message propagation.

```
if vertex v receives message m over edge e then
    v forwards m over all incident edges e' with γ(e') < γ(e).
end if
```

(a) For each $n \in \mathbb{N}$ show that there is a graph $H$ on $n$ vertices, a mapping $\gamma : E(H) \to \mathbb{N}$, a vertex $u \in V(H)$ and an edge $e$ incident to $u$ such that the algorithm above runs for at least $(n-1)^2/2$ rounds on $H$ when vertex $u$ starts the algorithm by sending a message along $e$.

(b) Now you would like to reduce the number of rounds this propagation algorithm takes to terminate by permuting $\gamma$. Prove the following lower bound on the time complexity you can achieve.

**Theorem.** *Let $\Delta$ denote the average degree of the graph $G$. For every injective edge labeling function $\gamma$ there exists a vertex $u$ and an edge $e$ incident to it such that if $u$ sends a message over $e$ then at least $\lfloor \Delta \rfloor$ rounds must pass until the process terminates.*

Note that this graded homework sheet counts 10% towards your final grade. As such this is considered as part of your exam and any attempt to cheat will be dealt with under ETH exam regulations. While you are allowed to talk to fellow students and consult books or the Internet, we expect you to hand in your own independent write-up. **Please name your pdf as your nethz username!**

Hand in your solutions as a LaTeX-typeset .pdf file (also attach your .tex file) by mail at `heinarss@inf.ethz.ch` or `rnenadov@student.ethz.ch` until Wednesday 24.4.2012, before 23:59.

**NOTE: Due to some misunderstanding we extend the deadline for exercise 4 (b) until Wednesday 1.5.2012. Note that in exercise 4 (b) you are supposed to prove a statement for all graps $G$, i.e. not only for the graph in part (a).**

[1] If a vertex sends a message in round $i$ the receiver will receive the message in round $i + 1$.