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Proposed solutions for graded HW 2

Graphs and Algorithms

Exercise 1 (Saturating Matching)

Note that it is enough to show that for every $S \subseteq A$ we have $|S| \leq |\Gamma(S)|$, as then the claim follows from the Hall's theorem.

We prove by induction on $|S|$ that this holds. For $|S| = 1$ we clearly have $|S| \leq |\Gamma(S)|$, as there are no isolated vertices in G. Assume that it holds for every $S' \subseteq A$ of size at most $n-1$, and consider some $S \subseteq A$, $|S| = n$. Let $a \in S$ and set $\bar{S} = S \setminus \{a\}$, and observe that by the induction hypothesis we have $|\bar{S}| \leq |\Gamma(\bar{S})|$. Without loss of generality, we may assume that $\Gamma(a) \subseteq \Gamma(\bar{S})$, as otherwise $|S| \leq |\Gamma(\bar{S})| + 1 \leq |\Gamma(S)|$ and we are done. We distinguish two cases. If $|\bar{S}| < |\Gamma(\bar{S})|$, then adding back a to $\bar{S}$ gives $|S| = |\bar{S}| + 1 \leq |\Gamma(\bar{S})| \leq \Gamma(S)$, and the condition holds.

Otherwise we have $|\bar{S}| = |\Gamma(\bar{S})|$. But now since for every subset $\bar{S}' \subseteq \bar{S}$ we have, by induction hypothesis, $|\bar{S}'| \leq |\Gamma(\bar{S}')|$, there exists a perfect matching between $\bar{S}'$ and $\Gamma(\bar{S}')$. Let us denote the edges of such matching by M, and for each edge $e \in M$ let us denote with e_v the endpoint of e that belongs to $\bar{S}$, and with e_u the endpoint which belongs to $\Gamma(\bar{S})$. Now by the exercise assumption we have

$$\sum_{e \in M} \deg(e_a) - \deg(e_b) \geq 0$$

and thus $\sum_{v \in \bar{S}} \deg(v) \geq \sum_{u \in \Gamma(\bar{S})} \deg(u)$. On the other hand, since $a \notin \bar{S}$ and there exists at least one edge between a and $\Gamma(\bar{S})$, we have

$$\sum_{v \in \bar{S}} \deg(v) < \sum_{u \in \Gamma(\bar{S})} \deg(u),$$

which leads to a contradiction. Therefore, the case $|\bar{S}| = |\Gamma(\bar{S})|$ cannot happen under the assumption $\Gamma(a) \subseteq \Gamma(\bar{S})$. Since in all other cases we have $|S| \leq |\Gamma(S)|$, this finishes the proof.

Exercise 2 (Brute forcing a lock)

Consider a graph where the vertices are digit strings of length 3. For each vertex on the form xyz where $x, y, z \in [9]$ we have a directed edge to all vertices of the form yzw where $w \in [9]$. Note that this graph is strongly connected since we can get from each vertex to any other by just "pressing" the digits of the other vertex. Also note that this graph is Eulerian because each vertex has out-degree 10 and in-degree 10. By first pressing three digits and then traversing the graph using an Eulerian tour we can use 10003 key presses to reach all combinations.

As it turns out this is also the minimum number of key presses we can do with in the worst case. To see this observe that each four digit string needs to appear in our sequence, each string can start at a single unique position and since there are 10000 possible strings we will need 10003 presses.

Exercise 3 (Minimum Degree Game)

Let us first consider the case where every vertex in G has an even degree. Then there exists a Eulerian tour, and let us orient the edges according to this tour. It is easy to see that every vertex has as many outgoing edges as ingoing, which is thus at least $2k$. Now the strategy of the first player is as following: whenever the second player colours an edge which is oriented from a vertex a to a vertex b, colour an arbitrary white edge which is an outgoing edge of a. Therefore, for every blue outgoing edge of any vertex a we have a corresponding outgoing red edge, thus there are at least k outgoing red edges for each vertex. This implies $\deg(G_r) \geq k$, and so it is a k-win for the player one!

Assume now that some vertices have an odd degree. Let us denote the set of all such vertices with V_o, and consider a graph G' obtained by adding a new vertex p to G, and connecting it to every vertex in V_o. Since $|V_o|$ is even, every vertex in G' has an even degree. By applying the same argument as in the previous case, there exists an orientation of edges in G' such that each vertex has at least $2k$ outgoing edges. However, in this case we have a stronger bound on the vertices in V_o. Since V_o has an odd degree in G, we have $\deg_G(v) \geq 4k + 1$ and thus $\deg_{G'}(v) \geq 4k + 2$, for each $v \in V_o$. This implies that there are at least $2k + 1$ outgoing edges associated with each vertex of V_o, and by removing the possible edge going to the vertex p, we have at least $2k$ outgoing edges. Now the same strategy as in the previous case yields a k-win for the player one.

Exercise 4 (Message propagation using edge ID's)

(a) We need to distinguish two cases, if n is odd or even.

Odd: If n is odd we consider K_n which has an Eulerian tour since the degrees of all vertices is even. For one specific Eulerian tour $e_1, e_2, \ldots, e_{|E|}$ we can assign the edge labels such that $\gamma(e_i) < \gamma(e_{i+1})$. Now if we start by sending a message over e_1 in the right direction the message will have to traverse the whole walk so the algorithm runs for at least $\binom{n}{2}$ many rounds.

Even: If n is even we again consider K_n. Now we cannot use the straightforward argument of an Eulerian path since the degrees are not even. However, note that since n is even there exists a perfect matching in the graph, let's consider the subgraph on all edges except the perfect matching. This subgraph has all degrees even so as before we can find an Euleria tour of length $\binom{n}{2} - \frac{n}{2}$. Note however that we could start the tour by traversing an edge of the matching since it's allowed to have two vertices of odd degree. We thus have by using that edge tour in the graph and the same way of labeling as in the odd case that the number of rounds for propagating the messages is at least $\binom{n}{2} - \frac{n}{2} + 1 = \frac{(n-1)^2}{2}$ which is the desired solution.

(b) For each vertex u in G denote by $d(u)$ the longest decreasing walk ending in u where it's decreasing in γ. Additionally denote by $d_i(u)$ the longest walk in u using only the i smallest edges w.r.t. γ. Observe that $d_0(u) = 0$ for all $u \in G$.

Now let s be the i-th smallest edge with endpoints u and v. We now have that $d_i(u) = d_{i-1}(v) + 1$ and $d_i(v) = d_{i-1}(u) + 1$ and for all other $w \in G$ we have that $d_i(w) \geq d_{i-1}(w)$. So for each i d_i increases by one for at least two vertices. We thus have that

$$\sum_{u \in G} d(u) \geq 2|E|$$

and thus

$$\frac{1}{|V|} \sum_{i \in G} d(u) \geq \frac{2|E|}{|V|} = \Delta.$$

Hence there must exist a vertex u such that $d(u) \geq \lfloor \Delta \rfloor$. Since we have proven the existence of a a decreasing walk of length at least Δ we have proven the theorem since if the first vertex on the walk sends a message across the first edge in the walk the message will need at least Δ rounds to traverse the whole walk (note that some nodes can receive the same message multiple times). $\square$