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FS 2013
Graded homework 3

Graphs and Algorithms

Exercise 1 (Nyan Colouring)

Let $G = (A, B, E)$ be a bipartite graph with partite sets A and B and minimum degree δ. Prove that the following algorithm *terminates* and returnts a colouring of edges of G such that for every vertex v and every colour $r \in [1, \delta]$ there exists an edge incident to v with colour r. Note that the colouring does not necessarily have to be proper.

To present the algorithm in a more concise way, we introduce some notation. For a colouring $c : E(G) \to [1, \delta]$, we say that a vertex $v \in V(G)$ is *non-rainbow* if it is not incident to all colours (i.e. there exists a colour which is not used by edges incident to v). For any colours $\alpha, \beta \in [1, \delta]$, we say that a walk W in G is a *(α, β)-walk* if colours of the edges along the walk alternate between α and β (starting with α). An (α, β)-walk is maximal if it cannot be extended.

Algorithm 1 Nyan Colouring.

Let $c : E(G) \to [1, \delta]$ be an arbitrary colouring
while there exists a non-rainbow vertex $v \in V(G)$ **do**
  $\beta \leftarrow$ a colour not incident to v in c
  $\alpha \leftarrow$ a colour used by at least two edges incident to v
  $W \leftarrow$ any maximal (α, β)-walk starting at v

$$c'(e) \leftarrow \begin{cases} c(e), & e \notin W \\ \beta, & e \in W \wedge c(e) = \alpha \\ \alpha, & e \in W \wedge c(e) = \beta. \end{cases}$$

  $c \leftarrow c'$
end while
return c

Exercise 2 (Line-Line graphs)

Prove that for a graph G it holds that $G = L(L(G))$ if and only if G is 2-regular.

Exercise 3 (Hamiltonian, Hamiltonian!)

Let G be a simple graph. Prove that if G is 3-regular and Hamiltonian then $L(G)$ can be decomposed into two edge disjoint Hamiltonian cycles.

Exercise 4 (Little Vizing)

Let $G = (V, E)$ be a graph on n vertices with maximum degree Δ such that $\Delta \geq 3$ and G contains no odd cycles of size larger than 4. Prove that $\chi'(G) = \Delta$.

The following **challenge exercise** *is not mandatory, and will not count towards your final grade. It is more difficult than the other exercises, and requires not only the understanding of the tools we have seen so far, but also creative thinking. Solving it will win you a book – see the webpage for the available book prices. But: only the* first *submitted correct solution will be awarded! You can submit your solution via email. Of course, it is an affair of honor that you only hand in solutions of your own.*

Exercise 5 (Challenge Exercise)

Let $k \in \mathbb{N}$, and consider a graph $G = (V, E)$ which satisfies the property that the number of edges in $G[V']$ is at most $k \cdot |V'|$, for every $V' \subseteq V$. Prove that the edges of G can be oriented such that the outdegree of every vertex is at most k.

Hint: One way to solve this problem is to construct a certain bipartite graph and use Hall's theorem to show that it has a saturating matching. Then deduce the orientation of edges from this matching.

NOTE THAT THIS GRADED HOMEWORK SHEET COUNTS 10% TOWARDS YOUR FINAL GRADE. AS SUCH THIS IS CONSIDERED AS PART OF YOUR EXAM AND ANY ATTEMPT TO CHEAT WILL BE DEALT WITH UNDER ETH EXAM REGULATIONS. WHILE YOU ARE ALLOWED TO TALK TO FELLOW STUDENTS AND CONSULT BOOKS OR THE INTERNET, WE EXPECT YOU TO HAND IN YOUR OWN INDEPENDENT WRITE-UP. **PLEASE NAME YOUR PDF AS YOUR NETHZ USERNAME!**

HAND IN YOUR SOLUTIONS AS A LATEX-TYPESET .PDF FILE (ALSO ATTACH YOUR .TEX FILE) BY MAIL AT `heinarss@inf.ethz.ch` or `rnenadov@inf.ethz.ch` until Wednesday 15.5.2013, before 23:59.