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Proposed solutions for graded HW 3

Graphs and Algorithms

Exercise 1 (solution by Anja Frei)

We start with an arbitrary colouring $c : E(G) \to [1, \delta]$ and assume that c is not yet a colouring as we want to have (otherwise we are already finished), which means that there exists a non-rainbow vertex v in G. So we want to look, what changes in the colouring when we execute the algorithm in the while-loop. We distinct four cases:

Case 1a W ends in a vertex $u \neq v$ and ends with an edge coloured with α.
In this case, the number of edges incident to u coloured with α is greater than the number of edges incident to u coloured with β (otherwise the (α, β)-walk could have been extended). So, by switching the colours on W, we do not decrease the number of colours incident to u. Also we increases the number of colours incident to v by 1. And last but not least, we remark that we didn't change the number of colours incident to the vertices in the middle of the walk and the ones not containd in W.

Case 1b W ends in a vertex $u \neq v$ and ends with an edge coloured with β.
In this case, the number of edges incident to u coloured with α is greater than the number of edges incident to u coloured with α (otherwise the (α, β)-walk could have been extended). So, by switching the colours on W, we do not decrease the number of colours incident to u. Also we increases the number of colours incident to v by 1. And last but not least, we remark that we didn't change the number of colours incident to the vertices in the middle of the walk and the ones not containd in W.

Case 2a W ends in v and ends with an edge coloured with α.
Since the walk starts with α and ends with α, we know that the length of the walk is odd. But this cannot happen, because G is a bipartite graph and a cycle in a bipartite graph has always even length.

Case 2b W ends in v and ends with an edge coloured with β.
This case cannot happen, because β by definition is a colour not incident to v.

We notice that we did not decrease the number of colours incident to any vertex during the while-loop, and we further observe that we increased the number of colours incident to v by 1. So, every time we execute the while-loop, we increases the number of incident colours for at least one vertex, but we never decrease it. Therefore, because the number of vertices and the number of colours and the number of edges per vertex are finite, the algorithm will terminate and will return the expected colouring.

Exercise 2 (solution by Barbara Geissmann)

We proof that $L(L(G)) = G \iff G$ is 2-regular.

"$\Longleftarrow$"

If G is 2-regular, then G is a collection of disconnected cycles. The line graph treats every connected component individual. Thus, it is enough to show that the line-line graph of a cycle is a cycle of the same size. Let C be a cycle on k vertices. C has the same number of vertices and edges. Therefore, $L(C)$ has the same number of vertices as C. Furthermore, the degree of every vertex in the $L(G)$ is 2, since every edge in C is adjacent to 2 other edges. Thus, the line graph is 2-regular as well. Since the line graph of a connected component is connected, $L(C)$ is connected as well. Therefore, $L(C)$ can only be a cycle and we have $C = L(C)$. This is true for all cycles in G, thus $L(G) = G$. From this we can conclude that $L(L(G)) = L(G) = G$. Notice that in fact $L^k(G) = G$ is true for any non-negative number k, given G 2-regular.

"$\Longrightarrow$"

Assume towards a contradiction that G is not 2-regular.

Case 1: $\exists v \in V$ with $\deg(v) = 0$
The line graph of a single vertex is the empty graph. The line graph of the empty graph is still the empty graph. Thus, v disappears. Since a line graph does never create new connected components, the number of connected components in $L(L(G))$ is smaller than the number in G. This is a contradiction to $L(L(G)) = G$.

Case 2: $\exists v \in V$ with $\deg(v) = 1$
Let $CH = (u, c_1, \ldots, c_k)$ be a "claw-chain" of maximum length in G starting in a vertex u, where $\deg(v) = 1$ and and c_k is the only other vertex that does not have degree 2. In the line graph CH consists of k vertices and $k-1$ edges. The vertex w corresponding to $\{c_{k-1}, c_k\}$ has degree 1 if $\deg(c_k) = 1$ and degree at least 3 if $\deg(c_k) \geq 3$. Thus, the chain in the line graph stops in w. We conclude that a claw-chain always shrinks when constructing the line graph of G. Since CH was of maximum length, there is no claw-chain of the same length in $L(G)$, and thus neither in $L(L(G))$. This is a contradiction to $L(L(G)) = G$.

Case 3: $\exists v \in V$ with $\deg(v) \geq 3$
We have that all vertices have degree ≥ 2, otherwise one of the two former cases applies. Let $|E_L|$ denote the number of edges in the line graph, $2|E_L| = \sum_{u \in V} (\deg(u) - 1) \deg(u)$. Let $|V_{LL}|$ denote the number of vertices in the line-line graph, $|V_{LL}| = |E_L|$. We get

$$|V_{LL}| \geq \frac{1}{2} \sum_{u \in V \setminus \{v\}} \deg(u) + \deg(v) > \frac{1}{2} \sum_{u \in V} \deg(u) = |V|.$$

This is a contradiction to $L(L(F)) = G$.

Exercise 3 (solution by Barbara Geissmann)

Let C be a Hamiltonian cycle in G. Let $= (e_1, e_2, \ldots, e_n)$ be the edges in C. Observe that the edges which are not in C form a perfect matching for the vertices in G. Thus, for every pair of edges (e_i, e_{i+1}) in C with a common endvertex v_i, there exists an edge m_j which is incident to v_i. This implies that in $L(G)$ the vertices representing e_i, e_{i+1}, and m_j build a triangle. Observe that every edge in G is part of exactly two such triangles, since G is 3-regular.

Now let us consider the line graph. We follow that $L(G)$ is Hamiltonian, since G is Hamiltonian, and $L(G)$ is 4-regular since G is 3-regular. Thus, 2 edge-disjoint Hamiltonian cycles in $L(G)$ will together use all edges. We give a constructive proof of how to find two edge-disjoint Hamiltonian cycles in $L(G)$.

The edges of C are now vertices. Let us denote them by e'_i. As shown before, there are at least two paths from an e'_i to an e'_{i+1}: The direct edge or the detour via m'_j. We do now the following greedy algorithm:

We follow the vertices e'_i in the same order as in C. And whenever the corresponding m'_j was not yet visited we take the detour to get from one vertex to the next. Otherwise, we take the direct link.

Since every m'_j is part of 2 such detours, it will certainly be included into the described cycle. Thus, all vertices are part of the first cycle. When we remove this first cycle from $L(G)$, a 2-connected graph is left. We need to show that this subgraph is connected. Observe that for every pair (e'_i, e'_{i+1}) either the detour or the direct link is in the subgraph. Therefore, all e'_i are contained in a cycle. Moreover, all m'_j are also contained in this cycle since for each m'_j one of its two detours is in the subgraph. Thus the subgraph forms another Hamiltonian cycle. Notice that we could as well run the greedy algorithm into the other direction of C and therefore get the second Hamiltonian cycle.

Exercise 4 (solution by Steve Muller)

Vizing's theorem from the lecture can be modified so that the edge chromatic number has to equal Δ, the maximum degree of a graph, under stronger assumptions. For the sake of completeness, we copy the unchanged parts of the theorem from the lecture notes.

Theorem. *Let G be a graph with maximum degree $\Delta \geq 3$ which does not contain odd cycles of size larger than 4. Then $\chi'(G) = \Delta$.*

Proof. We construct a colouring recursively (over amount of edges). Let $G = (V, E)$ be a graph which satisfies the conditions in the statement. Take any edge $e = \{u, v_0\} \in E$ and set $G' := (V, E \setminus e)$. Obviously we do not create cycles by removing edges.

Case 1 ($\Delta(G') = 2$) By Vizing's theorem we get a colouring of 3 colours. Three situations can occur.

1. There is an available colour for e. Then we are done.
2. e is adjacent to three edges, all of which have distinct colours.
3. e is adjacent to four edges, three of which have distinct colours.

For the last two cases, without loss of generality u has two neighbours. Denote the incident edges at u by a_1 and a_2, those at v_0 by b_1 and potentially b_2. In the third case we would have $c(b_2) = c(a_2)$, see figure 1, in the second case the edge b_2 would not exist.

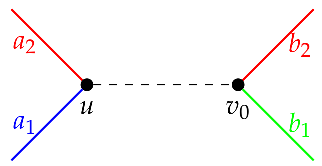


Figure 1: Situation where there is no free colour for edge $\{u, v_0\}$.

Set $R := c(a_2) = c(b_2)$, $G := c(b_1)$, $B := c(a_1)$.

Case 1.a a_1 *intersects* b_1. Note that then both of them do not intersect any further edge (because $\Delta(G') = 2$) $(*)$.

Look at the (unique) maximal R-G-alternating path P starting at u. It cannot[1] contain b_2 because then P stops at b_1 because of $(*)$ (hence $|P| \geq 4$) and we would get a cycle of odd length (P combined with a_1) larger than 4. So we can swap the colours on P such that $c(a_2) \leftarrow$ G without creating conflicts. If the edge b_2 exists, swap the colour on a maximal R-B-alternating path (which cannot contain a_1 by $(*)$). By consequence, we can colour e with R, which has just become available.

Case 1.b a_1 *does not intersect* b_1.

Look at the (unique) maximal B-G-alternating path P starting at u. By assumption P cannot contain b_1 because then $|P| \geq 4$ and we would get an odd cycle (P together with e) of length larger than 4. Swapping the colours on P makes B available at u so we can colour e with B.

Case 2 ($\Delta(G') \geq 3$) By the induction hypothesis we get a colouring c of G' which uses Δ colours. Since $\deg_{G'}(u) = \deg_G(u) - 1 \leq \Delta - 1$ there is one colour missing at u – say a_0.

Build a maximal sequence $(v_0, a_0), \ldots, (v_k, a_k)$ such that

- $a_i := c(\{u, v_i\})$
- $v_0, \ldots, v_k$ are distinct neighbours of u in G
- a_{i+1} is a colour missing at v_i, for any $0 \leq i < k$

In contrast to Vizing's theorem, there are *three* reasons why the sequence stops at v_k.

1. There is no colour missing at v_k – that is, $\deg v_k = \Delta$.
2. If a_{k+1} is a colour missing at v_k:
   (a) a_{k+1} is missing at u.
   (b) $\exists 1 \leq l \leq k \quad a_{k+1} = a_l$.

As for reasons 2.1 and 2.2, they work exactly the same as in Vizing's theorem. Indeed, they only rely on the structure of the sequence, and not on the amount of colours we have (Δ instead of $\Delta + 1$ as in Vizing's theorem). We copy them for completeness.

[1] Assuming b_2 exists. If it does not, P certainly does not contain it either.

Reason 2.1 a_{k+1} is missing at u. Then recolour $\{v_k, u\}$ with a_{k+1} and downshift the colours $a_k, \ldots, a_0$.

Reason 2.2 $\exists 1 \leq l \leq k \quad a_{k+1} = a_l$. Then a_{k+1} is missing at v_k and at v_{l-1}. If a_0 is missing at v_k, then we can colour $\{u, v_k\}$ with a_0 and downshift the colours $a_k, \ldots, a_1$.

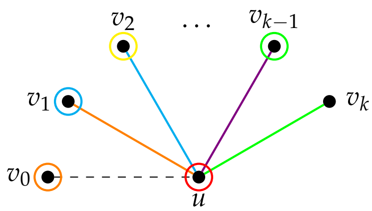
So assume a_0 appears at v_k. Let P be the (unique) maximal path of edges coloured alternatingly with $a_{k+1} = a_l$ and a_0, starting in v_k with a_0.

If P contains v_l, it must end in u. Then swap colours $a_0|a_l$ in P and downshift $a_l, \ldots, a_1$.

If P contains v_{l_1}, it must end in v_{l-1}. Then swap colours $a_0|a_l$ in P so that a_0 becomes available at v_{l-1}. We can then colour $\{v_{l-1}, u\}$ with a_0 and downshift $a_{l-1}, \ldots, a_1$.

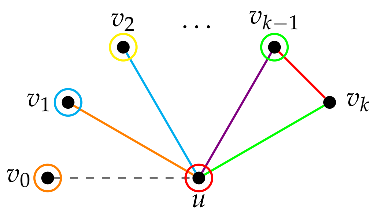
If P neither contains v_l nor v_{l_1}, swap colours $a_0|a_l$ in P so that a_0 becomes available at v_k. We can then colour $\{v_k, u\}$ with a_0 and downshift $a_k, \ldots, a_1$.

Reason 1 No colour is missing at v_k. In the picture below, the circles indicate which colour is missing at the respective vertex.

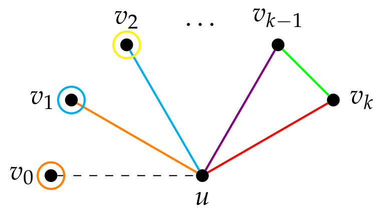


Let P be the (unique) maximal path of edges coloured alternatingly with $\textcolor{green}{\bullet}\, a_k$ and $\textcolor{red}{\bullet}\, a_0$, starting in u with $\textcolor{green}{\bullet}\, a_k$. If P does not contain v_{k-1}, then we can swap the colours $\textcolor{green}{\bullet}\, a_k \,|\, \textcolor{red}{\bullet}\, a_0$ in P so that $c(\{u, v_k\}) = \textcolor{red}{\bullet}\, a_0$, and downshift $a_k, \ldots, a_1$ to get the colouring we want.

So suppose P contains v_{k-1}. Since $\textcolor{green}{\bullet}\, a_k$ is missing at v_{k-1}, P must end in v_{k-1} with colour $\textcolor{red}{\bullet}\, a_0$. Then $|P| \in 2\mathbb{N}$ and hence there is a cycle (P combined with $\{u, v_{k-1}\}$) of odd length. By assumption, there are no cycles of odd length larger than 4, so there must be an edge $x = \{v_k, v_{k-1}\}$ with $c(x) = \textcolor{red}{\bullet}\, a_0$.



By assumption, $c(\{v_k, v_{k-1}\}) = \textcolor{red}{\bullet}\, a_0$ is missing at u and $c(\{v_k, u\}) = \textcolor{green}{\bullet}\, a_k$ is missing at v_{k-1}, so we can just swap them without creating any conflicts.



Now let Q be the (unique) maximal path of edges coloured alternatingly with ● a_{k-1} and ● a_k, starting in u with ● a_{k-1} – Q then continues with v_k, therefore[2] $|Q| \geq 3$. Note that Q cannot contain v_{k-2} for if it would, it would have to stop there with ● a_k (since ● a_{k-1} is missing), so $|Q| \in 2\mathbb{N}$ and we would have an odd cycle (Q together with $\{v_{k-1}, u\}$) of size larger than 4.

Swapping the colours ● a_{k-1} | ● a_k in Q makes ● a_{k-1} available at u and downshifting $a_{k-1}, \ldots, a_1$ gives the colouring we want. $\square$

[2]Recall that all colours, in particular ● a_{k-1}, are present at v_k.