ETH Zürich
Institute of Theoretical Computer Science
Dr. J. Lengler
H. Einarsson, R. Nenadov

FS 2013
Proposed solutions for sheet 3

Graphs and Algorithms

Solutions

Exercise 1 (Small and regular)

We prove that the graph on Figure 1 is a minimal 3-regular graph with connectivity 1.

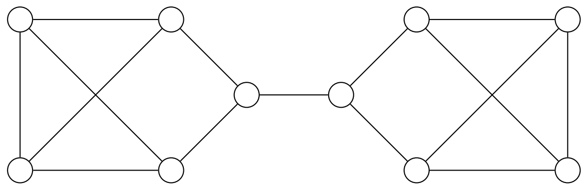

Figure 1: 3-regular graph with connectivity 1.

Consider an arbitrary 3-regular graph $G = (V, E)$ with connectivity 1. By the definition it has an articulation point $v \in V$, and since it is 3-regular there are either 2 or 3 connected components in $G \setminus v$. Moreover, in every connected component of $G \setminus v$ there exist either one or two vertices with degree 2, and all others have degree 3. On the other hand, there has to be an even number of vertices of degree 3 in every component of $G \setminus v$, and further there has to be at least 4 vertices in each component. Since the graph on Figure 1 satisfies all these minimal constraints, we conclude that there doesn't exist a 3-regular graph with less vertices which has connectivity 1.

Exercise 2 (Long Cycles)

For $k \geq 2$, let G be a k-connected graph on $n \geq 2k$ vertices. We want to show that G contains a cycle of length at least $2k$. As $k \geq 2$ there has to be some cycle in G.

Let C be the longest cycle in G and suppose the length l of C was less than $2k$. Then there is a vertex v not on the cycle. Define sets $A = \Gamma(v)$ and $B = V(C)$ (not necessarily disjoint).

Then $|A| \geq k$ and $3 \leq |B| \leq 2k - 1$. From Menger's theorem we obtain that there are $\min(k, |B|)$ disjoint A-B-paths. As the length of C is at most $2k - 1$, at least two of these paths must have endpoints that are neighbors on C by the pigeonhole principle. Let these two paths be $P = (a, \dots, b)$ and $P' = (a', \dots, b')$ where $a, a' \in A$ and $b, b' \in B$.

Now, we distinguish two cases. If $v \notin V(P) \cup V(P')$, we can create a new cycle by taking out the edge $\{b, b'\}$ from C and adding the paths P, P' and the two edges connecting v with those paths. Even though P and P' might contain only one vertex, the new cycle $(C - bb') + P + av + va' + P'$ is of length at least $l - 1 + 2 = l + 1$.

If one of the paths P, P' contains v, say $v \in P'$, let P'' be the path that goes from b' to v along P'. Now, again taking out the edge $\{b, b'\}$ from C and this time adding P, P'' and the edge $\{a, v\}$ gives a cycle $(C - bb') + P + av + P''$. Because v is not in C, P'' consists of at least one edge. Hence, the new cycle has length at least $l - 1 + 2 = l + 1$. In both cases the choice of C as a cycle of maximal length is contradicted.

The graph $K_{k,k}$, that is, the complete bipartite graph with parts of size k, shows that the statement of the theorem is best possible (the graph is k-connected and the longest cycle has length $2k$).

Exercise 3 (k Connectivity of the Hypercube)

A vertex $(a_1, a_2, \ldots, a_k)$ of Q_k has neighbors $(a_1, a_2, \ldots, a_{i-1}, \bar{a_i}, a_{i+1}, \ldots, a_k)$, $1 \leq i \leq k$, where $\bar{0} = 1$ and $\bar{1} = 0$. Thus, Q_k is k-regular and $\kappa(Q_k) \leq k$.

To see that $\kappa(Q_k) \geq k$ we proceed by induction on k. Q_1 is an edge and obviously connected. For $k \geq 2$, suppose that $S \subset V(Q_k)$ with $|S| \leq k - 1$ separates Q_k. Let $V_i = \{x \in V(Q_k) | x_k = i\}$ for $i = 0, 1$. Then $Q^0 = Q_k(V_0)$ and $Q^1 = Q_k(V_1)$, the induced subgraphs on these vertex sets, are both isomorphic to Q_{k-1}. Let furthermore $M = \{\{(x, 0), (x, 1)\} | x \in \{0, 1\}^{k-1}\}$ such that $E(Q_k) = E(Q^0) \cup E(Q^1) \cup M$.

If S lies completely in either V_0 or V_1, say V_0, then Q^1 is still connected and all remaining vertices in Q^0 are connected to this side by their matching edges. If S has vertices in both V_0 and V_1, both copies of Q_{k-1} are still connected by induction hypothesis and as $2^{k-1} - (k - 2) > 2^{k-2}$, there are still more than half of the vertices on both sides. Thus, there is also a remaining matching edge connecting both sides.

Exercise 4 (Subgraph with large minimum degree)

Proof of part (1). Heart of the proof is the following procedure: set $V' = V$, and as long as there exists a vertex $v \in V'$ with degree less than $\frac{d}{2} \cdot n$, remove v from V'. Observer that $G[V']$, where V' is the resulting subset, satisfies the minimal degree condition.

Let us estimate size of the subset V'. First, observe that during the process we have removed at most

$$\sum_{\ell=|V'|+1}^{n} \frac{d}{2} \cdot n = \frac{(n - |V'|) \cdot dn}{2}$$

edges. On the other hand in the remaining subgraph $G[V']$ there are at most $\frac{|V'|^2}{2}$ edges. Since the graph G has at least dn^2 edges, and each edge is either removed or belongs to the resulting subgraph, we have

$$\frac{(n - |V'|) \cdot dn}{2} + |V'|^2/2 \geq dn^2,$$

and thus by ignoring the $-|V'|dn/2$ term we get

$$|V'| \geq \sqrt{d} \cdot n,$$

which proves the claim. $\square$

Proof of part (2). First, we can assume that $dn > 1$, as otherwise any edge satisfies conditions of the exercise.

Consider again the procedure described in the proof of part (1), however this time removing vertices of degree less than dn. Note that, to prove the exercise, it suffices to show that $|V'| > 0$ once the procedure stops. Let us assume the opposite and observe that in the last two steps we have removed at most 1 edge. Thus the total number of removed edges is at most

$$\sum_{\ell=3}^{n} dn + 1 = (n-3) \cdot dn + 1 = dn^2 - 3dn + 1.$$

On the other hand, in this case we have that all edges are removed, and therefore

$$dn^2 - 3dn + 1 \geq dn^2,$$

which implies $1 > dn$ – contradiction! Therefore, the procedure has stopped before reaching $|V'| = 0$, thus $G[V']$ satisfies the claim.

Note: a more direct proof by induction on the number of vertices is also possible.

$\square$