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Proposed solutions for sheet 4

Graphs and Algorithms

Solutions

Exercise 1 (Vertex vs. edge connectivity)

For a given $t \in \mathbb{N}$, consider a graph $G = (V, E)$ on $2t + 1$ vertices (labeled from 1 to $2t + 2$), such that $G[\{1, \dots, t + 1\}]$ and $G[\{t + 1, \dots, 2t + 1\}]$ are complete graphs on $t + 1$ vertices, and further there is no edge between vertices $a, b \in [2t+1]$ such that $a < t+1$ and $b > t+1$ (or the other way around). Clearly, the vertex with label $t + 1$ is an articulation point, thus the vertex connectivity of G is 1. We show that edge-connectivity of G is t.

To prove that edge-connectivity of G is t, it suffices to prove that edge-connectivity of K_{t+1}, a complete graph on $t + 1$ vertices, is t. To this end, suppose that $t - 1$ edges are removed from a K_{t+1}, and consider two endpoints, say vertices u and v, of one such removed edge. If there exists a vertex w, different from u and v, such that both $\{u, w\}$ and $\{w, v\}$ are not removed, then u and v belong to the same connected component. Since there are $t - 1$ such possible vertices w, and at most $t - 2$ edges removed (since we already assumed that $\{u, v\}$ is removed), we conclude that at least one such vertex exists. This proves that K_{t+1} stays connected after removal of an arbitrary set of at most $t - 1$ edges.

Exercise 2 (Exam question, 2010.)

Let $G = (V, E)$ be a graph which satisfies condition of the exercise, and let $X \subseteq V$ be any subset of size at most $k - 1$. Idea of the proof is to show that any two vertices $u, v \in V \setminus X$ are either directly connected or there exists a vertex $z \in V \setminus X$ such that $\{u, z\}, \{z, v\} \in E$. Note that this implies k-connectivity of G.

Let $u, v \in V \setminus X$ be two distinct vertices, and assume that $\{u, v\} \notin G$. By an inclusion-exclusion principle, we have

$$|\Gamma(u) \cup \Gamma(v)| = |\Gamma(u)| + |\Gamma(v)| - |\Gamma(u) \cap \Gamma(v)|,$$

and thus $|\Gamma(u) \cap \Gamma(v)| = \deg(u) + \deg(v) - |\Gamma(u) \cup \Gamma(v)|$. On the other hand we assumed that $\{u, v\} \notin E$, thus $u \notin \Gamma(v)$ and $v \notin \Gamma(u)$, which implies an easy upper bound $|\Gamma(u) \cup \Gamma(v)| \leq n - 2$. Therefore,

$$\begin{aligned} |\Gamma(u) \cap \Gamma(v)| &\geq \deg(u) + \deg(v) - n + 2 \\ &\geq n + k - 2 - n + 2 = k, \end{aligned}$$

hence $(\Gamma(u) \cap \Gamma(v)) \setminus X \neq \varnothing$. In other words, there exists $z \in V \setminus X$ such that z is connected to both u and v.

Exercise 3 (The k-dimensional grid)

Let's assume that $n \geq 2$ since for $n = 1$ the k dimensional grid is a single vertex. We have that the k dimensional grid is at most k-connected since the vertex $(0, \ldots, 0)$ has degree k and removing its neighbors will disconnect the graph.

We show that for every pair of vertices $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ there exist at least k disjoint paths between them. This implies, by Menger's theorem, that the k-dimensional grid is k-connected. First let us assume that the coordinates are all different, i.e. $x_i \neq y_i$ for $i \in [k]$. We'll construct the k disjoint paths by fixing the coordinates, the first path we consider is

$$P_1 = (x_1, x_2, \ldots, x_k) \cdots (y_1, x_2, \ldots, x_k) \cdots (y_1, y_2, \ldots, x_k) \cdots (y_1, y_2, \ldots, y_k),$$

i.e. we first fix coordinate one, then the second and so on. We construct paths $P_2, \ldots, P_k$ in the same way except for path P_i we start by fixing coordinate i, then $i + 1$ and after fixing coordinate k we fix the first coordinate, then second and so on. We now claim that any two such paths will never intersect at an intermediate vertex. If there exists a vertex v such that paths P_i and P_j for $i \neq j$ both traverse v then it must hold that at vertex v both paths have fixed the same set of coordinates, but by our construction this cannot hold.

Now let's consider the case if x and y agree in some coordinates. Let's assume that they agree in $t < k$ coordinates and differ in $k - t$ coordinates. Then by the above construction we can construct $k - t$ disjoint paths by fixing coordinates. Now if $x_i = y_i$ we can create a new path by first changing coordinate i arbitrarily then use any of the $k - t$ coordinate fixing paths and then in the end step we fix coordinate i again. This way we can create at least t more disjoint paths totaling with at least k disjoint paths and thus the k-dimensional grid is k-connected.

Exercise 4 (Dilworth's theorem)

Let $(P, \leq)$ be a finite partially ordered set, and consider an antichain $A \subseteq P$. Then for any chain $C \subseteq P$ we have $|A \cap C| \leq 1$. Otherwise, there would exist two distinct elements $a, b \in A \cap C$ such that neither $a \leq b$ nor $b \leq a$, but then by the definition of a chain they cannot both belong to C – thus a contradiction. Therefore, we need at least $|A|$ chains to cover elements of A, hence at least $|A|$ chains to cover P.

Let us now prove that there exists an antichain A an a set of chains $\mathcal{C} = \{C_1, \ldots, C_t\}$ which covers P, such that $|\mathcal{C}| \leq |A|$. Consider a bipartite graph $G = (P_1 \cup P_2, E)$, as suggested in the hints. First, observe that if $V_c \subseteq P_1 \cup P_2$ is a vertex cover of G, then $P \setminus V_c$ is an antichain. If it's not an antichain, then there exist two vertices $a \in P_1$ and $b \in P_2$ such that $\{a, b\} \in E$ and $a, b \in P \setminus V_c$. But this implies that the edge $\{a, b\}$ is not covered by V_c, thus a contradiction with the fact that V_c is a vertex cover. Let us now consider a smallest vertex cover $V_c \subseteq P_1 \cup P_2$, and an antichain A that it induces. Since some vertices in V_c might correspond to the same element of P, the antichain A has at least $|P| - |V_c|$ elements. On the other hand, for any maximum matching $M \subseteq E$ in G, by Kőnig's theorem, we have $|M| = |V_c|$. We now create the family of chains by including a and b in the same chain whenever $\{a, b\} \in M$, where $a \in P_1$ and $b \in P_2$. It is easy to see that such sets are indeed chains. If a and b end up in the same set, then there has to exists a set of vertices $\{a, x_1, \ldots, x_k, b\}$

such that $\{a, x_1\}, \{x_1, x_2\}, \ldots, \{x_k, b\} \in M$. Let us assume that $a \leq x_1$. Then we also have $x_1 \leq x_2$, as otherwise we would have edges $\{a, x_1\}, \{x_2, x_1\} \in M$ such that $a, x_2 \in P_1$ and $x_1 \in P_2$, which cannot be because M is a matching. Repeating the same argument, and using transitivity, we get $a \leq b$. Similar argument shows that if $x_1 \leq a$ then $b \leq a$. Since every edge in M joins two chains (left as an exercise to the reader), and in the beginning every element of P is a chain for itself (since we want to have a set of chains that cover P), we end up with $|P| - |M|$ chains that cover P. From $|M| = |V_c|$ we conclude that there are at most as many chains as there are elements in A.