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Proposed solutions for sheet 5

Graphs and Algorithms

Exercise 1 (Matchings in Trees)

Assume for the sake of contradiction that there are two different perfect matchings M_1 and M_2 of a tree T and consider the symmetric difference $M := (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$. Note that in general such a symmetric difference of two matchings consists only of even cycles and paths. Since T is a tree and $M \neq \varnothing$ we have at least one path in M. The end vertex of such a path is only matched in one of the two matchings which is a contradiction to our assumption.

Exercise 2 (Line Graph of Regular Graphs)

It is easy to see that every vertex v in $L(G)$ is of degree $2(r-1)$. It remains to show that $L(G)$ is connected. Consider two vertices in $L(G)$, corresponding to the edges $e_1 = \{u_1, v_1\}$ and $e_2 = \{u_2, v_2\}$ in G, respectively. Now since G is connected there is a path $u_1 = w_1, w_2, \ldots, w_r = u_2$ and thus $\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{r-1}, w_r\}$ form a path in $L(G)$ where the first and last vertex are either identical to e_1 and e_2 or adjacent to them, respectively.

Exercise 3 (Knights on the Chessboard)

The answer is 32. A moment's reflection shows that placing 32 knights on the white squares (or the black squares) is a legal constellation (note that a single knight's move always leads from a white to a black square and vice versa). Why can we not place more than 32 knights on the board? Let F be the graph whose vertices are the 64 fields of the chessboard, and two vertices are connected if and only if the corresponding fields can be reached from each other by a single knight's move. The given problem translated into the graph theoretic setting asks for a maximum independent set in F. Note that if U is a (maximum) independent set in any graph G, then $V(G) \setminus U$ is a (minimum) vertex cover in G. Observe also that in any graph, the size of a (maximum) matching is a lower bound for the size of a (minimum) vertex cover (each vertex covers at most one edge from the matching). Now suppose there was an independent set U in F of size larger than 32. Then $V(F) \setminus U$ is a vertex cover in F of size smaller than 32. But it is well-known that F has a Hamilton cycle, implying that F has a perfect matching of size 32 (take every second edge from the Hamilton cycle). This contradicts the existence of a vertex cover of size smaller than 32.

Note: The complement of a vertex cover is an independent set and vice-versa.

Exercise 4 (Fast Matching Approximation - Exam 2012)

If you recall the Hopcroft & Karp's algorithm it finds in $O(\sqrt{n})$ rounds a maximum matching where in each round it finds an inclusion-maximal set of shortest augmenting paths and each round requires $O(m)$ steps.

Thus if only a constant number of augmentation rounds suffice the total running time of the algorithm will still be $O(m)$ and we are done. We will show that this is the case.

First observe that by Fact 6 on the slides (lecture 5+6) we will increase the length of the augmenting paths in each round since in each round we find a maximal set of disjoint shortest augmenting paths, and thus afterwards every shortest augmenting path must be strictly larger by Fact 5. Also observe that the length of an augmenting path is always odd so in fact the length of augmenting paths increases by at least two in each round.

By Fact 3, if all augmenting paths in matching M are of length $\geq t$ and if M' is a maximum matching then we have that

$$|M'| \leq |M| + \frac{n}{t}.$$

In order to get a .99-approximation it suffices that $\frac{n}{t} \leq 0.01 \cdot |M'|$. This is the case for $t \geq 400$, since then

$$0.01 \cdot |M'| \geq 0.01 \cdot \frac{n}{4} = \frac{n}{400} \geq \frac{n}{t}.$$

Since t is increased by at least 2 in each round, the condition $t \geq 400$ will be satisfied after 200 rounds of the Hopcroft-Karp algorithm. These 200 rounds can be performed in time $200 \cdot O(m) = O(m)$, as desired.