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Proposed solutions for sheet 6

Graphs and Algorithms

Exercise 1 (Hamiltonian Graphs)

We proceed similarly as in the proof of the Dirac's theorem.

First we check that the graph is connected. Consider any two vertices $u, v \in V(G)$, and suppose that there is no edge between them. Then $|\Gamma(v) \cap \Gamma(u)| = \deg(v) + \deg(u) - |\Gamma(v) \cup \Gamma(u)| \geq n - (n-2)$, thus u and v have a common neighbour. Therefore there exists a path between any two vertices in G.

Let us now assume that the graph is not Hamiltonian, and consider a longest path P in G. Let $v_1, \ldots, v_t$, where $t = |P|$, be the vertices of P such that there is an edge between v_i and v_{i+1} for every $1 \leq i \leq t-1$. By the assumption that P is the longest path, we have $\Gamma(v_1) \subseteq P$ and $\Gamma(v_t) \subseteq P$. Let us now define sets S_1 and S_2 as following,

$$S_1 = \{i \in [t-1] \mid \{v_1, v_{i+1}\} \in E(G)\}$$
$$S_2 = \{i \in [t-1] \mid \{v_i, v_t\} \in E(G)\}.$$

Since $|S_1| = \deg(v_1)$ and $|S_2| = \deg(v_t)$, we have

$$|S_1 \cap S_2| = \deg(v_1) + \deg(v_2) - |S_1 \cup S_2| \geq n - t.$$

On the other hand, from assumption that G is not Hamiltonian we have $|T| \leq n-1$, thus $S_1 \cap S_2 \neq \varnothing$. Let us consider any $z \in S_1 \cap S_2$, and observe that

$$C = (v_1, v_{z+1}, v_{z+2}, \ldots, v_t, v_z, v_{z-1}, \ldots, v_2, v_1)$$

forms a cycle of length $t < n$. However, the fact that G is connected implies that there exists an edge between a vertex from C and a vertex from $V(G) \setminus C$, hence we can construct a path which is by at least one longer than the path P – a contradiction! Therefore G has to be Hamiltonian.

Exercise 2 (Hamilton Paths in Tournaments)

We argue by induction over the number of vertices. The basis for the induction is trivial. Now let v be any vertex of a tournament T_n, and $N^{\leftarrow}$ and $N^{\rightarrow}$ the set of in-neighbors and out-neighbors of v, respectively. As $T_n - v$ is a tournament on $n-1$ vertices, we can apply the induction hypothesis and obtain a Hamilton path H from some vertex x to another vertex y in $T_n - v$. If $x \in N^{\rightarrow}$, then $vx + P$ is a Hamilton path in T_n. If $y \in N^{\leftarrow}$, then $P + yv$ is a Hamilton path in T_n. Otherwise we have $x \in N^{\leftarrow}$ and $y \in N^{\rightarrow}$, implying that there is an edge $z_1 z_2$ on P with $z_1 \in N^{\leftarrow}$ and $z_2 \in N^{\rightarrow}$. Then $P - z_1 z_2 + z_1 v + v z_2$ is a Hamilton path in T_n.

With a similar induction one can show that there is exactly one Hamilton path in an acyclic tournament. Since the tournament is acyclic it contains a sink, thus all Hamiltonian paths must end in the sink. Removing the sink will give another acyclic tournament on $n-1$ vertices which also contains a sink. Thus there is only a single Hamiltonian path in an acyclic tournament.

Exercise 3 (Adjacency Matrix)

As we saw in the lecture, if A is the adjacency matrix of a graph G, then the element $a_{i,j}$ of A^k, for some $k \in \mathbb{N}$, is exactly the number of walks of length k from i to j.

Observe that any walk of length 3, which starts and finishes in the same vertex v, is neccesarily a triangle. Additionaly, for each triangle there are exatly 3 starting vertices, and each can *tour* the triangle in two directions. Thus the number of triangles in G equals to

$$\frac{1}{6} \cdot \sum_{i=1}^{n} a_{i,i} = tr(A^3)/6,$$

where $A^3 = (a_{i,j})_{n \times n}$.

Let us now count the number of cycles of length 4. Similarly as in the previous case, every cycle of length 4 can be represented as two (distinct) walks of length 2 between vertices u and v. Since for every cycle we have 2 possible choices for pairs of opposite vertices, the number of cycles of length 4 equals to

$$\frac{1}{2} \cdot \sum_{i<j} \binom{a_{i,j}}{2},$$

where $A^2 = (a_{i,j})_{n \times n}$.