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FS 2013
Proposed solutions for sheet 7

Graphs and Algorithms

Exercise 1 (Greedy is good, greedy is bad)

(a) Since G can be colored with $\chi(G)$ take any such coloring c of G. Now consider a permutation such that all the vertices of color 1 in c are listed first (in any order). After them in the permutation we list all vertices of color 2 and so on.

Now in a run of the greedy coloring algorithm each vertex in a permutation as described above will receive either the same color as in c or a lower valued color. It cannot receive a larger color since no two neighbors have the same color in c and hence the color of the vertex given with c will always be available.

(b) We construct an infinite family of rooted trees T_k ($k \geq 0$) as follows: Let T_0 be an isolated vertex and T_k the tree that is obtained by connecting a new root vertex v to the root vertices of copies of all trees $T_0, T_1, \ldots, T_{k-1}$. Clearly we have $n(T_k) = 2^k$. We want to show that there is an ordering π of the vertices of T_k such that $\textsc{Greedy-Coloring}(T_k, \pi) = k + 1 = \Omega(\log n)$. Such an ordering π can be defined recursively. T_0 has only one vertex and there is only one possible ordering. The ordering of the vertices of T_k is defined as follows: First visit the vertices of T_0, then those of T_1 etc. up to T_{k-1}, and finally visit the new root vertex v (the vertices in each subtree are visited in the order defined for this subtree). It is easy to see by induction that using this ordering the greedy coloring algorithm will always need an additional $(k + 1)$-th color to color the new root vertex v.

Exercise 2 (Directed edge coloring)

Consider the following bipartite graph $G' = (A, B, E')$ where the partite sets A and B are copies of $V(G)$ and for $u \in A$ and $v \in B$ we have that $\{u, v\} \in E'$ if and only if $(u, v) \in E(G)$ (i.e. if there is a directed edge from u to v).

Now we have by construction that $\Delta(G') \leq k$. Thus by König's Corollary to Vizing's theorem we can find a k edge coloring of G'. This edge coloring now corresponds to an edge coloring in G with the desired property since each edge incident to a vertex in A corresponds to out-going and each edge incident to a vertex in B corresponds to in-going edges which will thus receive pairwise distint colors.

Exercise 3 (Girth six, $\chi(G) = k$)

Let's first show that H has girth 6. It is evident that the girth cannot be greater than 6 since H contains copies of G as subgraphs and G has a cycle of length 6 since its girth is 6.

Now let's assume for the sake of contradiction that $C = v_1, v_2, \dots, v_r$ with $r \leq 5$ is a cycle in H. Since G has girth 6 it must be that one of the vertices in C is in T, let's assume that $v_1 \in T$. We now have that the vertices in T only have a single neighbour in each copy of G it is connected to so it must be that v_2 and v_r are in different copies G_1 and G_2 of G respectively. But now since v_2 and v_r both only have a single neighbour in T it must be that $v_3 \in G_1$ and $v_{r-1} \in G_2$ but now v_3 and v_{r-1} can't be connected since G_1 and G_2 are not connected which is a contradiction to our assumption.

Let's show that $\chi(H) = \chi(G) + 1$. We can easily see that $\chi(H) \leq \chi(G) + 1$ since we can color each copy of G independently with k colors and then color T with a new color.

Now let's assume for the sake of contradiction that H is k colorable. Now since T contains nk vertices we have by the pigeon-hole principle that there exists a subset S with n vertices in T which is colored with the same color. Now let G be the copy which is connected to S, it must be that G is colored with $k - 1$ colors since the color of S would not be allowed in G, which is a contradiction since $\chi(G) = k$. Thus $\chi(H) = k + 1$ as desired.