1 Trees

1. A δ-regular tree is a tree where all inner vertices have degree δ. Show that there is a δ-regular tree with n vertices if and only if $n \geq \delta + 1$ and if $n \equiv 2 \pmod{(\delta - 1)}$.

2. Show that if $n \equiv 2 \pmod{(\delta-1)}$, the number of labelled δ-regular trees is $\binom{n}{k}\binom{n-2}{\delta-1,\delta-1,\ldots,\delta-1}$ where the multinomial coefficient $\binom{m}{k_1,k_2,\ldots,k_t}$ is defined as $\binom{m}{k_1,k_2,\ldots,k_t} := \frac{m!}{k_1!k_2!\cdots k_t!}$.

2 Planar Graphs

1. Show that the complement of a simple planar graph with n vertices is non-planar for $n \geq 11$.

2. Let G be a planar graph with $n \geq 3$ vertices and $3n - 6$ edges embedded in the plane.

   (a) Show that all faces of G are triangles.

   (b) Show that if G has chromatic number 3, then it is Eulerian.

3 Connectivity

1. Let $G = (V, E)$ be a k-connected graph and let D be the diameter of G. Show that $|V| \geq k(D-1)+2$ and that the size of the largest independent set of G $\alpha(G) \geq \lceil (D+1)/2 \rceil$.

2. Let $G = (V, E)$ be a graph with minimum degree $\delta(G) \geq |V|/2 + t$ for $0 \leq t < |V|/2 - 1$. Show that G is $(2t + 2)$-connected.

3. Let G be a graph with n vertices such that any two distinct vertices x and y satisfy $\deg(x) + \deg(y) \geq n - 1$. Prove that G is connected.

4. For each $n \geq 2$ give an example for a disconnected graph on n vertices such that any two distinct vertices x and y satisfy $\deg(x)+\deg(y) \geq n-2$.

4 Coloring

Let k and n be natural numbers with $k \geq 1$ and $n \geq k(k + 1)$. Place n points on a circle and let $G_{n,k}$ be the $2k$-regular graph obtained by joining each point to the k nearest points in each direction on the circle. For example $G_{n,1}$ is a cycle on n vertices.

1. Prove that $\chi(G_{n,k}) = k+1$ if $k+1$ divides n and otherwise $\chi(G_{n,k}) = k+2$.

2. Show that the lower bound on n cannot be weakened by proving that $\chi(G_{k(k+1)-1,k}) > k + 2$ if $k \geq 2$.