Eidgenössische
Technische Hochschule
Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Federal Institute of Technology at Zurich

Institut für Theoretische Informatik
Peter Widmayer
Thomas Tschager

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Datenstrukturen & Algorithmen — Exercise Sheet 1 — FS 16

There is a definition of the $\mathcal{O}$ notation that is different from the one given at the lecture. Namely, for a function $g : \mathbb{N} \to \mathbb{R}^+$, let

$$\mathcal{O}(g) := \{f : \mathbb{N} \to \mathbb{R}^+ \mid \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N},\ \forall n \geq n_0 : f(n) \leq cg(n)\}. \tag{1}$$

Analogously, we say that a function f grows asymptotically at least as much as g, if $f \in \Omega(g)$ with

$$\Omega(g) := \{f : \mathbb{N} \to \mathbb{R}^+ \mid \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N},\ \forall n \geq n_0 : f(n) \geq cg(n)\}. \tag{2}$$

A function f grows asymptotically like g when $f \in \mathcal{O}(g)$ and $f \in \Omega(g)$. We denote this by $f \in \Theta(g)$, or as $f = \Theta(g)$.

For these exercises, you can choose to use the definition given at the lecture, or use the above definition.

Exercise 1.1 *The Set $\Theta(g)$.*

Give a counterexample that demonstrates that the right-hand side of the following equation does *not* hold:

$$\Theta(g) = \left\{f : \mathbb{N} \to \mathbb{R}^+ \;\middle|\; \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N},\ \forall n \geq n_0 : f(n) = cg(n)\right\}.$$

Give a correct definition of the set $\Theta(g)$ as compactly as possible (i.e., with the fewest possible parameters and quantifiers), analogously to the above definitions for the sets $\mathcal{O}(g)$ and $\Omega(g)$.

Exercise 1.2 *Proofs about $\mathcal{O}$ Notation.*

Prove or disprove the following statements, where $f, g : \mathbb{N} \to \mathbb{R}^+$.

a) $f \in \mathcal{O}(g)$ if and only if $g \in \Omega(f)$.

b) If $f \in \mathcal{O}(g)$, then $f(n) \leq g(n)$ for every $n \in \mathbb{N}$.

c) If $f(n) \leq g(n)$ for every $n \in \mathbb{N}$, then $f \in \mathcal{O}(g)$.

d) There exist different functions f and g such that $f \in \Omega(g)$ and $g \in \Omega(f)$.

e) $\log_a(n) \in \Theta(\log_b(n))$ for all constants $a, b \in \mathbb{N}\backslash\{1\}$.

f) Let $f_1, f_2 \in \mathcal{O}(g)$ and $f(n) := f_1(n) + f_2(n)$. Then, $f \in \mathcal{O}(g)$.

g) Let $f_1, f_2 \in \mathcal{O}(g)$ and $f(n) := f_1(n) \cdot f_2(n)$. Then, $f \in \mathcal{O}(g)$.

h) $n^{1/a} \in \Theta(n^{1/b})$ for all $a, b \in \mathbb{N}$, $a \leq b$,

Please turn over.

Exercise 1.3 *Asymptotic Growth of Functions.*

Sort the following functions from left to right such that: if function f is to the left of g, then $f \in \mathcal{O}(g)$.

Example: the functions n^3, n^7, n^9 are already in the right order since $n^3 \in \mathcal{O}(n^7)$ and $n^7 \in \mathcal{O}(n^9)$.

$$n^5 + n,\ \log(n^4),\ \sqrt{n},\ \binom{n}{3},\ 2^{16},\ n^n,\ n!,\ \frac{2^n}{n^2},\ \log^8(n)$$

Hand-in: Wednesday, 2nd March 2016 in your exercise group.