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Solutions to Handout 10

Complex Network Models

Solution to Exercise 1

Let u be a vertex of weight w_u and position x_u, and let v be a vertex of weight w_v. If $w_u w_v \geqslant n$, then u, and v are connected with probability 1 as required. We now assume $w_u w_v \leqslant n$. u and v connect if $||x_u - x_v||_\infty \leqslant (w_u w_v)^{\frac{1}{d}}$. This probability is:

$$\Theta(1) \int_0^{(w_u w_v)^{\frac{1}{d}}} \frac{r^{d-1}}{n} dr = \Theta\left(\frac{1}{n}\right) \int_0^{w_u w_v} dr = \Theta\left(\frac{w_u w_v}{n}\right)$$

We've shown that

$$\mathbb{P}[u \sim v \mid w_u, x_u, w_v] = \Theta\left(\min\left\{1, \frac{w_u w_v}{n}\right\}\right)$$

Solution to Exercise 2

1. First, we note that there are in expectation $\Theta(n^2)$ pairs of vertices that could *potentially* be connected by an edge. The upper bound is obvious. For the lower bound, consider that for any given pair, with constant probability one vertex chooses a position in C_1 and the other in C_2. Therefore, it remains to compute the probability of an edge between two such vertices.

   Since $u \in C_1$ and $v \in C_2$, we have $\|u - v\| = \Theta(L)$. Hence, conditioning on the two endpoints lying in the two cubes, the connection probability is

   $$\mathbb{P}(u \sim v) = \mathbb{E}_{w_u, w_v}\left[\min\left\{1, \left(\frac{w_u w_v}{\Theta(L^d)}\right)^\alpha\right\}\right].$$

   As $L^d = \Theta(n)$, this is

   $$\mathbb{P}(u \sim v) = \Theta\Big(\mathbb{E}_{w_u, w_v}\left[\min\left\{1, \left(\frac{w_u w_v}{n}\right)^\alpha\right\}\right]\Big).$$

   Using the weight density $f(w) = \Theta(w^{-\tau})$ for $w \geq 1$, this becomes

   $$\mathbb{P}(u \sim v) = \Theta\Big(\int_1^\infty \int_1^\infty \min\left\{1, \left(\frac{w_u w_v}{n}\right)^\alpha\right\} w_u^{-\tau} w_v^{-\tau}\, dw_u\, dw_v\Big).$$

   Let

   $$I_n := \int_1^\infty \int_1^\infty \min\left\{1, \left(\frac{w_u w_v}{n}\right)^\alpha\right\} w_u^{-\tau} w_v^{-\tau}\, dw_u\, dw_v.$$

   We split according to whether $w_u w_v \leq n$:

   $$I_n = n^{-\alpha} \int_1^n \int_1^{n/w_u} (w_u w_v)^\alpha w_u^{-\tau} w_v^{-\tau}\, dw_v\, dw_u + \int_1^\infty \int_{\max\{1, n/w_u\}}^\infty w_u^{-\tau} w_v^{-\tau}\, dw_v\, dw_u.$$

   For the first term,

   $$I_n^{(1)} = n^{-\alpha} \int_1^n w_u^{\alpha-\tau} \int_1^{n/w_u} w_v^{\alpha-\tau}\, dw_v\, dw_u.$$

If $\alpha < \tau - 1$, then the integrals are bounded by a constant, so

$$I_n^{(1)} = \Theta(n^{-\alpha}).$$

If $\alpha > \tau - 1$, then

$$\int_1^{n/w_u} w_v^{\alpha-\tau}\, dw_v = \Theta\left(\left(\frac{n}{w_u}\right)^{\alpha-\tau+1}\right),$$

and therefore

$$I_n^{(1)} = \Theta\left(n^{-\alpha}\int_1^n w_u^{\alpha-\tau}\left(\frac{n}{w_u}\right)^{\alpha-\tau+1} dw_u\right) = \Theta\left(n^{1-\tau}\int_1^n \frac{dw_u}{w_u}\right) = \Theta\left(n^{1-\tau}\log n\right).$$

For the second term,

$$I_n^{(2)} = \int_1^\infty \int_{\max\{1,n/w_u\}}^\infty w_u^{-\tau} w_v^{-\tau}\, dw_v\, dw_u.$$

We split at $w_u = n$:

$$I_n^{(2)} = \int_1^n w_u^{-\tau}\int_{n/w_u}^\infty w_v^{-\tau}\, dw_v\, dw_u + \int_n^\infty w_u^{-\tau}\int_1^\infty w_v^{-\tau}\, dw_v\, dw_u.$$

For the first part (since $\tau > 1$),

$$\int_{n/w_u}^\infty w_v^{-\tau}\, dw_v = \Theta\big((n/w_u)^{1-\tau}\big),$$

so

$$\int_1^n w_u^{-\tau}(n/w_u)^{1-\tau}\, dw_u = n^{1-\tau}\int_1^n w_u^{-1}\, dw_u = \Theta\big(n^{1-\tau}\log n\big).$$

For the second part,

$$\int_1^\infty w_v^{-\tau}\, dw_v = \Theta(1),$$

hence

$$\int_n^\infty w_u^{-\tau}\, dw_u = \Theta\big(n^{1-\tau}\big).$$

Combining both contributions,

$$I_n^{(2)} = \Theta\big(n^{1-\tau}\log n\big).$$

Finally, we get

$$\mathbb{P}(u \sim v) = \Theta(I_n) = \begin{cases} \Theta(n^{-\alpha}), & \alpha < \tau - 1, \\ \Theta(n^{1-\tau}\log n), & \alpha > \tau - 1. \end{cases}$$

Therefore,

$$\mathbb{E}[X] = \begin{cases} \Theta(n^{2-\alpha}), & \alpha < \tau - 1, \\ \Theta(n^{3-\tau}\log n), & \alpha > \tau - 1. \end{cases}$$

2. We can see that as soon as $\alpha > 2$ and $\tau > 3$ we have $\mathbb{E}[X] = o(1)$ so Markov's inequality tells us that in this regime $X = 0$ with high probability.

   We will now show that if one of $\alpha < 2$ or $\tau < 3$ holds, we have $X \geq 1$ with high probability. This does not automatically follow from the expectation being large[1], but in this case we can show that it does. Let us start with the easier case $\alpha < 2$. Now, regardless of the weights of the vertices, each of the $\Theta(n^2)$ pairs *independently* has a probability at least $\Theta(n^{-\alpha})$ to connect. Therefore X

[1] Such mistakes have been made in the literature and they are surprisingly easy to make, so one should always be careful.

is dominated by a binomial distribution with $\Theta(n^2)$ trials and $\Theta(n^{-\alpha})$ success probability. This implies that with high probability X is at least 1.

For the case $\tau < 3$ we will show a similar statement with a bit more care. Now we cannot simply say that different edges are independent, because using the weight information about one might influence the other (if they share a vertex). However, we can still argue as follows. Since $\tau < 3$, the number of vertices in C_1 with weight at least $\sqrt{n}$ follows a binomial distribution with $\omega(1)$ expectation. So with high probability we have at least polynomially many such vertices in C_1. Similarly for C_2. Now, conditional on these vertices existing, each pair again has an *independent* constant probability to connect, as the product of the endpoint weights is at least n. Another argument regarding binomial distributions finishes the proof.

Remark. As an example of a false proof misled by the expectation being large, notice that we would still have an expected $\Theta(n^{3-\tau})$ number of edges of the type where one endpoint has weight at least n and the other has constant weight. But with high probability vertices of the former type do not exist, as their expectation is only $\Theta(n^{2-\tau}) = o(1)$! In the very rare event that one does exist, the expected number of edges gets a big "cashout"; namely roughly n edges, which leads to the $n^{3-\tau}$ term in the overall expectation.