ETH Zurich
Institute of Theoretical Computer Science
Prof. Dr. Johannes Lengler
Konstantinos Lakis

FS 2025
Solutions to Handout 11

Complex Network Models

Solution to Exercise 1

Let $x > 1$.

$$\begin{aligned}\mathbb{P}[XY = x] &= \int_1^x \frac{\mathbb{P}[X = u]\mathbb{P}[Y = x/u]}{u} du \\ &= \Theta(1) \int_1^x \frac{u^{-\tau} \left(\frac{x}{u}\right)^{-\tau}}{u} du \\ &= \Theta(1) x^{-\tau} \int_1^x \frac{du}{u} \\ &= \Theta(1) x^{-\tau} \log x\end{aligned}$$

Solution to Exercise 2

- There are in expectation $\Theta(R^d)$ vertices in each of V_s and $V_b \setminus V'$, and the distance between a pair of vertices from $V_s \times V_b \setminus V'$ is $\Theta(R)$. For one such pair, there is a strong tie if $w_u w_v \geqslant d(u,v)^d$, which happens with probability

$$\Theta(1) \int_{d(u,v)^d}^{+\infty} x^{-\tau} \log x dx = \tilde{\Theta}(d(u,v)^{d(1-\tau)}) = \tilde{\Theta}(R^{d(1-\tau)})$$

  There is a weak tie with probability

$$\Theta(1) \int_1^{d(u,v)^d} x^{-\tau} \log x \frac{x^\alpha}{d(u,v)^{\alpha d}} dx = \Theta(d(u,v)^{-\alpha d}) \Theta(1) = \Theta(R^{-\alpha d})$$

  The overall probability is $\tilde{\Theta}(R^{d(1-\tau)}) + \Theta(R^{-\alpha d}) = \tilde{\Theta}(R^{-\mu d})$. Counting all such pairs, we get in expectation $\Theta(R^2 d)\tilde{\Theta}(R^{-\mu d}) = \tilde{\Theta}(R^{(2-\mu)d})$ edges.

- Let $\mathcal{X}''$ be the cube centered in p of radius $R - 1/4$. There are $\Theta(R^{d-1})$ vertices in $\mathcal{X}' \setminus \mathcal{X}''$. Let v be such a vertex. Let $B(v,1)$ be the ball centered on v of radius v. This ball intersects $\mathcal{X}_b \setminus \mathcal{X}'$ in area of constant volume. There are in expectation a constant number of vertices in that area, and all of them are connected to v, since their distance to v is smaller than 1. In expectation, we have thus at least $\Theta(R^{d-1}) = \Theta(R^{d(1-1/d)})$ edges between $\mathcal{X}'$ and $\mathcal{X}_b$.