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Solutions to Handout 2

Complex Network Models

Solution to Exercise 1

Given a vertex, the probability that it is in a component of size 3 is:

$$\begin{aligned}\mathbb{P}[v \text{ in a component of size 3}] &= \mathbb{P}\left[\bigcup_{\{a,b\}\in\binom{V\setminus\{v\}}{2}} \{v,a,b\} \text{ is a component of size 3}\right]\\ &= \binom{n-1}{2}(1-p)^{3(n-3)}(p^3+3p^2(1-p))\end{aligned}$$

By linearity of expectation:

$$\mathbb{E}[n_3] = n\binom{n-1}{2}(1-p)^{3(n-3)}(3p^2(1-p)+p^3)$$

where $(1-p)^{3(n-3)}$ is the probability that the 3 vertices are disconnected from the rest, and $(3p^2(1-p)+p^3)$ is the probability that the three vertices are connected, i.e. the probability that the number of edges (which is binomially distributed) is 2 or 3.

Let X_k be the random variable counting the number of children of the $(k-1)$th discovered node. For a Galton-Watson tree to be of size 3, there are two possibilities:

$$\begin{aligned}p_3 &= \mathbb{P}[X_0=2]\mathbb{P}[X_1=0]\mathbb{P}[X_2=0] + \mathbb{P}[X_0=1]\mathbb{P}[X_1=1]\mathbb{P}[X_2=0]\\ &= e^{-3\mu}\mu^2\left[\frac{1}{2}+1\right] = \frac{3}{2}e^{-3\mu}\mu^2\end{aligned}$$

We easily confirm that $\frac{\mathbb{E}[n_3]}{n} \to p_3$

Similar computation yield to:

$$\mathbb{E}[n_4] = n\binom{n-1}{3}(1-p)^{4(n-4)}(16p^3-33p^4+24p^5-6p^6)$$

and

$$p_4 = \frac{8}{3}e^{-4\mu}\mu^3$$

The former can, for instance, be obtained by first considering three vertices. They either form 1, 2, or 3 components. In each case, we need at least one edge from the last vertex to each of the components. The computation is not immediate, but shows all the relevance of the theorem.

Solution to Exercise 2

Let's fix $\mu > 1$ and $s \in \mathbb{N}$. There are a lot of possible distributions. One such choice is the following:

- if $s = 1$, $\mathbb{P}[\mathcal{D} = 0] = 1$
- else if $\mu \leqslant s - 1$, then

$$\begin{aligned}\mathbb{P}[\mathcal{D} = 0] &= 1 - \frac{\mu}{s-1} \\ \mathbb{P}[\mathcal{D} = s-1] &= \frac{\mu}{s-1}\end{aligned}$$

- else

$$\begin{aligned}\mathbb{P}[\mathcal{D} = 0] &= \frac{1}{s} - \frac{\mu - (1-\frac{1}{s})}{2s\lceil\mu\rceil} \\ \mathbb{P}[\mathcal{D} = 1] &= 1 - \frac{1}{s} \\ \mathbb{P}[\mathcal{D} = 2s\lceil\mu\rceil] &= \frac{\mu - (1-\frac{1}{s})}{2s\lceil\mu\rceil}\end{aligned}$$

In the last case, we have $p_t \geqslant \mathbb{P}[\mathcal{D} = 1]^{s-1}\mathbb{P}[\mathcal{D} = 0] = (1-\frac{1}{s})^{s-1}\mathbb{P}[\mathcal{D} = 0] \geqslant (1-\frac{1}{s})^{s-1}\frac{1}{2s} \geqslant \frac{e}{2s}$ where we used the inequality

$$\left(1 - \frac{1}{s}\right)^{s-1} > e$$

Solution to Exercise 3

Let us compute the probability that U and V intersect. We assume $k \leqslant \frac{n}{2}$ (otherwise, the two sets intersect necessarily). We have:

$$\begin{aligned}\mathbb{P}[U \cap V = \varnothing] &= \mathbb{P}[V \in [n] \setminus U] \\ &= \frac{\binom{n-k}{k}}{\binom{n}{k}} \\ &= \frac{(n-k)!(n-k)!}{(n-2k)!n!} \\ &= \frac{n-k}{n} \cdots \frac{n-2k+1}{n-k+1} \\ &= \left(1 - \frac{k}{n}\right) \cdots \left(1 - \frac{k}{n-k+1}\right)\end{aligned}$$

$$1 - \frac{2k^2}{n} \leqslant \left(1 - \frac{2k}{n}\right)^k \leqslant \left(1 - \frac{k}{n-k+1}\right)^k \leqslant \mathbb{P}[U \cap V = \varnothing] \leqslant \left(1 - \frac{k}{n}\right)^k \leqslant e^{-\frac{k^2}{n}}$$

where we used the two inequalities:

$$\forall x \in \mathbb{R}, \forall n \in \mathbb{N}, 1 + x \leqslant e^x$$

and

$$\forall \varepsilon \geqslant -2, \forall n \in \mathbb{N}, (1+\varepsilon)^n \geqslant 1 + n\varepsilon$$

If $k = o(\sqrt{n})$, then $k \leqslant \frac{n}{2}$ for n sufficiently large, and $\mathbb{P}[U \cap V = \varnothing] \geqslant 1 - o(1)$.

Similarly, if $k = \omega(\sqrt{n})$, then $\mathbb{P}[U \cap V = \varnothing] \leqslant o(1)$.