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Solutions to Handout 3

Complex Network Models

Solution to Exercise 1

Let ε such that $\mu = 1+\varepsilon$. Let $p' = \frac{1+\varepsilon/2}{n}$, and p'' such that $(1-p')(1-p'') = 1-p$, i.e. $p'' = \frac{p-p'}{1-p'}$. For n sufficiently large, we have $p'' \geqslant \frac{\varepsilon}{4n}$, so $p'' = \Omega\left(\frac{1}{n}\right)$

We obtain the same distribution by sampling first $G' = ([n], E') \sim G_{n,p'}$, then $G'' = ([n], E'') \sim G_{n,p''}$ and returning $G = ([n], E' \cup E'')$. We know that, w.h.p. G' has a path of length at least Cn, for some constant $C > 0$ independent of n. Let A and B be the first and last $0.1Cn$ vertices of the path. The probability that no edge is added between A and B in the second step is $(1-p'')^{(0.1Cn)^2} \leqslant e^{-p''(0.1Cn)^2} = e^{-\Omega(n)} = o(1)$.

We conclude that w.h.p., G will have a cycle of length at least $0.8Cn$, i.e. of linear length.

Solution to Exercise 2

1. For one vertex $v \in V$, the probability that it has a degree of one is:

$$\mathbb{P}[v \text{ has degree } 1] = (n-1)p(1-p)^{n-2} \to \mu e^{-\mu}$$

   By linearity of expectation, $\mathbb{E}[N_1] = n(n-1)p(1-p)^{n-2} = \Theta(n)$

2. (a) We have

$$2\mathbb{E}[|E_n|] = \mathbb{E}\left[\sum_{d=1}^{n-1} dN_d\right] = \sum_{d=1}^{n-1} d\mathbb{E}[N_d] \geqslant \sum_{d=1}^{n-1} d\frac{n}{d^\tau} = n\sum_{d=1}^{n-1}\frac{1}{d^{\tau-1}}$$

   If $\tau < 2$, then $\tau - 1 < 1$ and $\mathbb{E}[|E_n|] = n\Theta(n^{2-\tau}) = \Omega(n^{3-\tau})$

   (b) if $\tau < 1$, then by question a, $\mathbb{E}[|E_n|] = \Omega(n^{3-\tau})$ with $3-\tau > 2$, which is impossible, since $\mathbb{E}[|E_n|] = \mathcal{O}(n^2)$

   (c) if $\tau < 2$, then by question a, $\mathbb{E}[|E_n|] = \Omega(n^{3-\tau})$ with $3-\tau > 1$, which implies $\mathbb{E}[|E_n|] = \omega(n)$. If $\tau = 2$, then $2\mathbb{E}[|E_n|] = n\sum_{d=1}^{n-1}\frac{1}{d} \sim n\log n = \omega(n)$

Solution to Exercise 3

We start by proving the following lemma: for any integer sequence k such that $k(n) \to \infty$

$$\mathbb{P}[\mathrm{Po}(\mu) \geqslant k] \sim e^{-\mu} \frac{\mu^k}{k!}$$

To see this, note that $\mathbb{P}[\mathrm{Po}(\mu) \geqslant k] \geqslant e^{-\mu}\frac{\mu^k}{k!}$. On the other hand, for sufficiently large n

$$\begin{aligned}\mathbb{P}[\mathrm{Po}(\mu) \geqslant k] &= \sum_{l=k}^{\infty} e^{-\mu}\frac{\mu^l}{l!} = e^{-\mu}\frac{\mu^k}{k!}\sum_{r=0}^{\infty}\frac{\mu^r k!}{(k+r)!} \\ &\leqslant e^{-\mu}\frac{\mu^k}{k!}\sum_{r=0}^{\infty}\left(\frac{\mu}{k+1}\right)^r = e^{-\mu}\frac{\mu^k}{k!}\frac{1}{1-\frac{\mu}{k+1}} \sim e^{-\mu}\frac{\mu^k}{k!}\end{aligned}$$

which proves our lemma.

Now, let $c > 0$ be a constant $c \neq 1$, and k an integer sequence with $k(n) \sim c\frac{\log n}{\log\log n}$. We have for $n \in \mathbb{N}$:

$$\begin{aligned}\mathbb{P}[X_{max} < k] &= \mathbb{P}[\mathrm{Po}(\mu) < k]^n && \text{by independence} \\ &= (1 - \mathbb{P}[\mathrm{Po}(\mu) \geqslant k])^n \\ &\leqslant \exp(-n\mathbb{P}[\mathrm{Po}(\mu) \geqslant k]) && \forall x \in \mathbb{R}, 1 + x \leqslant e^x\end{aligned}$$

Also, by union bound

$$\mathbb{P}[X_{max} \geqslant k] \leqslant n\mathbb{P}[\mathrm{Po}(\mu) \geqslant k].$$

Observe that $n\mathbb{P}[\mathrm{Po}(\mu) \geqslant k]$ represents the expected number of occurrences of events with a weight greater than or equal to k. This quantity appears unsurprisingly in both expressions as a threshold indicator.

Applying the lemma, we get that

$$n\mathbb{P}[\mathrm{Po}(\mu) \geqslant k] \sim e^{-\mu} n\frac{\mu^k}{k!}$$

and by taking the logarithm,

$$\log\left(n\frac{\mu^k}{k!}\right) = \log n + k\log\mu - \log k!$$

Since

$$\log k! \sim k\log k \sim c\frac{\log n}{\log\log n}\log\log n = c\log n$$

and

$$k\log\mu = o(\log n)$$

we obtain

$$\log\left(n\frac{\mu^k}{k!}\right) \sim (1-c)\log n$$

For $c < 1$, $\log\left(n\frac{\mu^k}{k!}\right) \to +\infty$, so $n\mathbb{P}[\mathrm{Po}(\mu) \geqslant k] \to +\infty$ and $\mathbb{P}[X_{max} < k] \to 0$,

and for $c > 1$, $\log\left(n\frac{\mu^k}{k!}\right) \to -\infty$, so $n\mathbb{P}[\mathrm{Po}(\mu) \geqslant k] \to 0$ and $\mathbb{P}[X_{max} \geqslant k] \to 0$.

For any $0 < c < 1 < C$, by union bound

$$\begin{aligned}\mathbb{P}\left[c\frac{\log n}{\log\log n} \leqslant X_{\max} \leqslant C\frac{\log n}{\log\log n}\right] &\geqslant 1 - \mathbb{P}\left[X_{\max} < c\frac{\log n}{\log\log n}\right] - \mathbb{P}\left[X_{\max} > C\frac{\log n}{\log\log n}\right] \\ &\geqslant 1 - \mathbb{P}\left[X_{\max} < \left\lceil c\frac{\log n}{\log\log n}\right\rceil\right] - \mathbb{P}\left[X_{\max} \geqslant \left\lfloor C\frac{\log n}{\log\log n}\right\rfloor\right] \to 1\end{aligned}$$

so w.h.p.

$$c\frac{\log n}{\log\log n} \leqslant X_{\max} \leqslant C\frac{\log n}{\log\log n}.$$