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Solutions to Handout 4

Complex Network Models

Solution to Exercise 1

Let $(u_k) = (\frac{1}{Zk^\tau})$ where Z is the normalizing constant. Consider the following distribtion

$$\forall k \in \mathbb{N}, \mathbb{P}[\mathcal{D} = k] = \begin{cases} u_{k-1} + u_k & \text{if k is even} \\ 0 & \text{else} \end{cases}$$

We have the following equivalence:

$$\sum_{i \geqslant k} u_i \sim \frac{k^{1-\tau}}{(\tau-1)Z}$$

We have that

$$\mathbb{P}[\mathcal{D} \geqslant 2k] = \sum_{i \geqslant 2k-1} u_i \sim \frac{(2k-1)^{1-\tau}}{(\tau-1)Z} \sim \frac{(2k)^{1-\tau}}{(\tau-1)Z}$$

and

$$\mathbb{P}[\mathcal{D} \geqslant 2k+1] = \sum_{i \geqslant 2k+1} u_i \sim \frac{(2k+1)^{1-\tau}}{(\tau-1)Z}$$

which shows that $\mathcal{D}$ follows a strict cumulative power-law distribution with exponent τ. However, it is clear that $\mathcal{D}$ cannot follow a strict or weak power law distribution since it is null for an infinity of indices.

Solution to Exercise 2

Let $n \in \mathbb{N}$, $d \in [n]$. Let $S_d^n \coloneqq \#\{k \in [n], u_k = d\}$ and $C_d^n \coloneqq \#\{k \in [n], u_k \geqslant d\}$.

We have $u_i = d \iff d \leqslant \left(\frac{n}{i}\right)^{\frac{1}{\tau-1}} < d+1 \iff \frac{n}{(d+1)^{\tau-1}} < i \leqslant \frac{n}{d^{\tau-1}}$ so

$$\frac{n}{d^{\tau-1}} - \frac{n}{(d+1)^{\tau-1}} - 1 \leqslant S_d^n \leqslant \frac{n}{d^{\tau-1}} - \frac{n}{(d+1)^{\tau-1}}$$

Similarly, we have

$$\frac{n}{d^{\tau-1}} - 2 \leqslant C_d^n \leqslant \frac{n}{d^{\tau-1}} - 1$$

Observe that

$$\frac{1}{d^{\tau-1}} - \frac{1}{(d+1)^{\tau-1}} = \frac{1}{d^{\tau-1}} \left(1 - \frac{1}{(1+\frac{1}{d})^{\tau-1}}\right) \underset{d\to\infty}{\sim} \frac{\tau-1}{d^\tau}$$

As all the elements are positive, let $c_1, c_2 > 0$ such that

$$\forall d \in \mathbb{N}, \frac{c_1}{d^\tau} \leqslant \frac{1}{d^{\tau-1}} - \frac{1}{(d+1)^{\tau-1}} \leqslant \frac{c_2}{d^\tau}$$

For $d \leqslant D(n) \coloneqq \left(\frac{c_1 n}{2}\right)^{\frac{1}{\tau}}$, we have that $\frac{c_1 n}{2d^\tau} \geqslant 1$ so

$$\frac{c_1 n}{2d^\tau} \leqslant \frac{c_1 n}{d^\tau} - 1 \leqslant S_d^n \leqslant \frac{c_2 n}{d^\tau}$$

We conclude that the sequence u is a $(c_1/2, c_2)$-power-law distribution with exponent τ and cut-off $D(n) = \left(\frac{c_1 n}{2}\right)^{\frac{1}{\tau}}$.

Similarly, we can show that the sequence u is a $(1/2, 1)$-cumulative power-law distribution with exponent τ and cut-off $D'(n) = \left(\frac{n}{4}\right)^{\frac{1}{\tau-1}}$.