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Solutions to Handout 5

Complex Network Models

Solution to Exercise 1

1. Let D be the weight distribution.
   - We have that $C_1 \leqslant \frac{\mathbb{P}[D=w]}{w^{-\tau}} \leqslant C_2$ for some C_1, C_2. This means

     $$\frac{C_1}{\tau-1}[w^{1-\tau}]_{w_1}^{w_0} = C_1 \int_{w_0}^{w_1} w^{-\tau} dw \leqslant \mathbb{P}[D \in [w_0, w_1]] \leqslant C_2 \int_{w_0}^{w_1} w^{-\tau} dw = \frac{C_2}{\tau-1}[w^{1-\tau}]_{w_1}^{w_0}$$

     If $C > 1$, $[w^{1-\tau}]_{w_1}^{w_0} = w_0^{1-\tau} - w_1^{1-\tau} = \Theta(w_0^{1-\tau})$, and $\mathbb{P}[D \in [w_0, w_1]] = \Theta(w_0^{1-\tau})$
   - We have that $C_1 \leqslant \frac{\mathbb{P}[D \geqslant w]}{w^{1-\tau}} \leqslant C_2$ for some C_1, C_2. This means

     $$C_1 w_0^{1-\tau} - C_2 w_1^{1-\tau} \leqslant \mathbb{P}[D \in [w_0, w_1]] = \mathbb{P}[D \geqslant w_0] - \mathbb{P}[D \geqslant w_1] \leqslant C_2 w_0^{1-\tau} - C_1 w_1^{1-\tau} \leqslant C_2 w_0^{1-\tau}$$

     Here, to achieve the same result, we need to have $\frac{C_2 w_1^{1-\tau}}{C_1 w_0^{1-\tau}} \leqslant 1 - \varepsilon$ for any $\varepsilon > 0$, which is the case if $\frac{C_2}{C_1} < C^{\tau-1}$, i.e. $C > \left(\frac{C_2}{C_1}\right)^{\frac{1}{\tau-1}}$
2. Let u be a neighbour of v. w_u follows a power-law distribution with cutoff $\frac{n}{w}$. Following the same approach as before, we have that $\mathbb{P}[w_u \in [w_0, w_1]] = \Theta(w_0^{2-\tau})$ if $C > 1$. The expected number of such edges is thus $\Theta(w w_0^{2-\tau})$

Solution to Exercise 2

We start by computing the expected number of edges in G. For two fixed vertices u and v, we have

$$\int_1^{\sqrt{n}} \int_1^{\sqrt{n}} \Theta(1) u^{-\tau} v^{-\tau} \frac{uv}{n} du dv \leqslant \mathbb{P}[u \sim v] \leqslant \int_1^{+\infty} \int_1^{+\infty} \Theta(1) u^{-\tau} v^{-\tau} \frac{uv}{n} du dv$$

so

$$\frac{1}{n}\left(\int_1^{\sqrt{n}} \Theta(1) u^{1-\tau} du\right)^2 \leqslant \mathbb{P}[u \sim v] \leqslant \left(\int_1^{+\infty} \Theta(1) u^{1-\tau} du\right)^2 \frac{1}{n}$$

The two bounds are $\Theta(\frac{1}{n})$, so there are in expectation $\binom{n}{2}\Theta(\frac{1}{n}) = \Theta(n)$ edges. Moreover, the number of edges is concentrated around its expectation by Chernoff.

Let e be an (ordered) edge selected uniformly at random[1].

$$\begin{aligned}\mathbb{P}[w_u = w \mid e_1 = u] &= \frac{\mathbb{P}[w_u = w]\mathbb{P}[e_1 = u \mid w_u = w]}{\mathbb{P}[e_1 = u]} \\ &= \frac{\Theta(w^{-\tau})\mathbb{P}[e_1 = u \mid w_u = w]}{\frac{1}{n}}\end{aligned}$$

[1] we ignore the event that no edge exists

If $w_u \leqslant n$, then w.h.p., we have $\Theta(n)$ edges, and u is connected to $\Theta(w_u)$ edges, so $\mathbb{P}[e_1 = u \mid w_u = w] = \Theta(\frac{w}{n})$

We can conclude:

$$\mathbb{P}[w_u = w \mid e_1 = u] = \Theta(w^{1-\tau}).$$

The above show that the degree/weight of the vertex follows a strong power law up to weight n.

Please note that this is different from the following: sample a Chung-Lu graph with n vertices and then put all weights of endpoints of all edges in a list and sort them. That means that each vertex weight will be written down as many times as the number of edges adjacent to it.

Now, with high probability this sequence of weights will have no value larger than $n^{1/(\tau-1)+\varepsilon}$ (recall that this is larger than the maximum weight we expect to see with n samples from a power law). Therefore, we could not hope for a larger cutoff than that if we consider the precise sequence of weights found like this and demand that the number of indices i such that at position i we have value k is proportional to $k^{-\tau}$. If such an interpretation is made (which was not the aim of the exercise but is not unnatural), then the cutoff is actually roughly $n^{1/\tau}$ for a strong power law and $n^{1/(\tau-1)}$ for a weak power law. This is because at these values for k the expectation of valid numbers (either equal to k or larger than k respectively) starts dropping below 1.