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Solutions to Handout 6

Complex Network Models

Solution to Exercise 1

1. Le u be the neighbour of v of maximum weight. We know that $w_u \geqslant w^{\frac{1}{\tau-2}-\varepsilon/2}$ w.h.p.. If $w_u > n$, then $N_u = \Omega(n)$ w.h.p.. Otherwise, $N_u = \Theta(w_u)$ w.h.p.. As $N_v^{(2)} \geqslant N_u$, we have shown that there exist C independent of n such that $N_v^{(2)} \geqslant Cw^{\frac{1}{\tau-2}-\varepsilon/2} \geqslant w^{\frac{1}{\tau-2}-\varepsilon}$ w.h.p.[1].

2. We know that w.h.p., $N^1(v) = \Theta(w_v)$ and $\max_{u\in N(v)} w_u \leqslant w_{\max} := w_v^{\frac{1}{\tau-2}+\varepsilon/4}$. Let us compute the expected neighbourhood size of a neighbour of v conditioned on its weight being smaller than $w_{\max}$:

$$\begin{aligned}
\mathbb{P}[w_u = w \mid u \sim v, w_u \leqslant w_{\max}] &= \frac{\mathbb{P}[w_u = w, u \sim v \mid w_u \leqslant w_{\max}]}{\mathbb{P}[u \sim v \mid w_u \leqslant w_{\max}]} \\
&= \frac{\mathbb{P}[w_u = w \mid w_u \leqslant w_{\max}]\mathbb{P}[u \sim v \mid w_u = w]}{\mathbb{P}[u \sim v \mid w_u \leqslant w_{\max}]} \\
&\leqslant \Theta(1)\frac{\mathbb{1}\{w \leqslant w_{\max}\}w^{-\tau}\frac{ww_v}{n}}{\mathbb{P}[u \sim v \mid w_u \leqslant w_{\max}]} = \Theta(1)\frac{\mathbb{1}\{w \leqslant w_{\max}\}w^{1-\tau}\frac{w_v}{n}}{\frac{w_v}{n}\int_1^{w_{\max}} w^{1-\tau}dw} \\
&= \Theta(1)\frac{\mathbb{1}\{w \leqslant w_{\max}\}w^{1-\tau}}{\int_1^{w_{\max}} w^{1-\tau}dw} = \Theta(1)\mathbb{1}\{w \leqslant w_{\max}\}w^{1-\tau}
\end{aligned}$$

The conditional expectation is

$$\mathbb{E}[w_u \mid u \sim v, w_u \leqslant w_{\max}] \leqslant \Theta(1)\int_1^{w_{\max}} w^{2-\tau}dw = \Theta(1)w_{\max}^{3-\tau} = \Theta(1)w_v^{\frac{3-\tau}{\tau-2}+\varepsilon(3-\tau)/4}$$

and

$$\mathbb{E}[N^1(u) \mid u \sim v, w_u \leqslant w_{\max}] \leqslant \Theta(1)w_v^{\frac{3-\tau}{\tau-2}+\varepsilon(3-\tau)/4} \leqslant w_v^{\frac{3-\tau}{\tau-2}+\varepsilon/4}$$

We want to bound $N^1(u)$ w.h.p.. Let L be the event $w_u \leqslant w_{\max}$

$$\begin{aligned}
\mathbb{P}\left[N^1(u) \geqslant w_v^{\frac{3-\tau}{\tau-2}+\varepsilon/2} \mid u \sim v\right] &\leqslant \mathbb{P}[L \mid u \sim v]\mathbb{P}\left[N^1(u) \geqslant w_v^{\frac{3-\tau}{\tau-2}+\varepsilon/2} \mid u \sim v, L\right] + \mathbb{P}\left[\bar{L} \mid u \sim v\right] \\
&\leqslant (1-o(1))\mathbb{P}\left[N^1(u) \geqslant w_v^{\varepsilon/4}\mathbb{E}\left[N^1(u) \mid L\right] \mid L\right] + o(1) \\
&\leqslant (1-o(1))\frac{1}{w_v^{\varepsilon/4}} + o(1) = o(1)
\end{aligned}$$

where we used Markov for the last step.

We can finally bound w.h.p. the neighbourhood size as follows:

$$\begin{aligned}
N^2(v) &\leqslant N^1(v) + \sum_{u\sim v} N^1(u) \\
&\leqslant \Theta(w_v) + \Theta\left(w_v w_v^{\frac{3-\tau}{\tau-2}+\varepsilon/2}\right) \\
&\leqslant w_v^{\frac{1}{\tau-2}+\varepsilon}
\end{aligned}$$

[1] the latter inequality holds for n sufficiently large, but that is not a problem since we want the statement to hold w.h.p.

Solution to Exercise 2

Let $\mathcal{D}$ be the outdegree distribution. Let $x \geqslant 1$. We have

$$\mathbb{P}\left[\max_v d_v \leqslant x\right] = (\mathbb{P}[\mathcal{D} \leqslant x])^n = \left(1 - \Theta(x^{1-\tau})\right)^n$$

Similarly, $\mathbb{E}\left[\sum_v \mathbb{1}\{d_v \geqslant x\}\right] = n\Theta(x^{1-\tau})$

The threshold seems to be around $w_{\max} = n^{\frac{1}{\tau-1}}$. Indeed, if $x = \omega(w_{\max})$, then

$$\mathbb{P}\left[\max_v d_v \leqslant x\right] \leqslant \exp -n\Theta(x^{1-\tau})$$

Conversely, if $x = o(w_{\max})$, we have that

$$\mathbb{P}\left[\max_v d_v \leqslant x\right] \geqslant 1 - n\Theta(x^{1-\tau}) \to 1$$

1. If $\tau > 2$, the threshold is $n^{\frac{1}{\tau-1}} = o(n)$, so there is no vertex of weight larger than n w.h.p..
2. if $\tau \in]2,3[$, then $n^{\frac{1}{\tau-1}} = \omega(\sqrt{n})$ and there is a vertex of weight at least $\sqrt{n}$ w.h.p..
3. if $\tau > 3$, then $n^{\frac{1}{\tau-1}} = o(\sqrt{n})$ and there is no vertex of weight at least $\sqrt{n}$ w.h.p..