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Solutions to Handout 7

Complex Network Models

Solution to Exercise 1

We condition everything on w_u. Let N be a neighbour obtained through a random edge connected to u. If u has no neighbour, then $N = \varnothing$. Let v be a vertex, and $w \geqslant 1$. We have

$$\begin{aligned}
\mathbb{P}[w_v = w | N = v] &= \frac{\mathbb{P}[w_v = w]\mathbb{P}[N = v | w_v = w]}{\mathbb{P}[N = v]} \\
&= \frac{\mathbb{P}[w_v = w]\mathbb{P}[N = v | w_v = w]}{\mathbb{P}[N \neq \varnothing]\mathbb{P}[N = v \mid N \neq \varnothing]} \\
&= \Theta(1)\frac{\mathbb{P}[w_v = w]\mathbb{P}[N = v | w_v = w]}{\mathbb{P}[N = v \mid N \neq \varnothing]} \\
&= \Theta(1)\frac{w^{-\tau}\mathbb{P}[N = v | w_v = w]}{\frac{1}{n}}
\end{aligned}$$

We now want to estimate $\mathbb{P}[N = v | w_v = w]$. Let's condition on the weights (w_s). Let X be the random variable counting the number of edges between u and v, and let Y count the remaining edges of u. X and Y are independent, and $X \sim \mathrm{Po}(\lambda)$, where $\lambda := \frac{w_u w_v}{n}$, while $Y \sim \mathrm{Po}(\mu)$, where $\mu := \frac{w_u \sum_{s \neq u,v} w_s}{n}$. By the strong law of large numbers, w.h.p., $\frac{\sum_{s \neq u,v} w_s}{n} \to \mathbb{E}[w_s]$ for any $s \in V$. All of this being defined, we have[1]

$$\mathbb{P}[N = v | w_v = w] = \mathbb{E}\left[\frac{X}{X+Y}\right]$$

We first upper-bound this expression:

$$\begin{aligned}
\mathbb{E}\left[\frac{X}{X+Y}\right] &= \mathbb{P}[X \geqslant 1]\mathbb{E}\left[\frac{X}{X+Y} \mid X \geqslant 1\right] \\
&\leqslant \mathbb{P}[X \geqslant 1]\mathbb{E}\left[\frac{X}{1+Y} \mid X \geqslant 1\right] \\
&= \mathbb{P}[X \geqslant 1]\mathbb{E}[X \mid X \geqslant 1]\mathbb{E}\left[\frac{1}{1+Y} \mid X \geqslant 1\right] \\
&= \mathbb{E}[X]\mathbb{E}\left[\frac{1}{1+Y}\right] \\
&= \lambda\frac{1 - e^{-\mu}}{\mu} = \Theta\left(\frac{\lambda}{\mu}\right) = \Theta\left(\frac{w_v}{n}\right)
\end{aligned}$$

We now lower-bound it. By Markov, with constant probability, $X \leqslant 2\mathbb{E}[X]$ and $Y \leqslant 2\mathbb{E}[Y]$. Using theorem 1.1, we also have for some constant $\eta < 1$,

$$\mathbb{P}\left[X \leqslant \frac{\lambda}{2}\right] \leqslant \eta^{\lambda}$$

[1] the following expression only makes sense when the denominator is not null. Otherwise, the probability is itself null.

We deduce that there exitst a constant C such that for $\lambda \geqslant C$, $\eta^\lambda \leqslant \frac{1}{4}$, so with constant probability, $\frac{\lambda}{2} \leqslant X \leqslant 2\lambda$. Overall, we get for $\lambda \geqslant C$:

$$\mathbb{E}\left[\frac{X}{X+Y}\right] \geqslant \Theta(1)\frac{\lambda/2}{2\lambda+2\mu} = \Theta\left(\frac{\lambda}{\lambda+\mu}\right)$$

We need to fix a cutoff on w_v for $\lambda+\mu$ to be a $\Theta(\mu)$. This happens for example, for $w_v \leqslant n$. Observe that the cutoff doesn't depend on w_u. In that case, we get

$$\mathbb{E}\left[\frac{X}{X+Y}\right] \geqslant \Theta\left(\frac{\lambda}{\mu}\right) = \Theta\left(\frac{w_v}{n}\right)$$

For $\lambda \leqslant C$, we rather use

$$\mathbb{E}\left[\frac{X}{X+Y}\right] \geqslant \mathbb{P}[X=1]\mathbb{E}\left[\frac{1}{1+Y}\right] = e^{-\lambda}\lambda\frac{1-e^{-\mu}}{\mu} = \Theta\left(\frac{\lambda}{\mu}\right) = \Theta\left(\frac{w_v}{n}\right)$$

Overall, we've shown that for $w \leqslant n$

$$\mathbb{P}[w_v = w | N = v] = \Theta(1)\frac{w^{-\tau}\frac{w}{n}}{\frac{1}{n}} = \Theta(w^{1-\tau})$$

Note: One can actually compute the exact value of $\mathbb{P}\left[N = v \mid w_v = w\right]$. One can show the remarkable following fact:

$$\mathbb{P}\left[N = v \mid w_v = w, N \neq \varnothing\right] = \mathbb{E}\left[\frac{X}{X+Y} \mid X+Y>0\right] = \frac{\lambda}{\lambda+\mu}$$

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Solution to Exercise 2

Let $n \geqslant 3$. Let $\mathcal{D}$ be the distribution of the degree of a random vertex. Let for $u, v \in [n]$, $X_{u,v}$ be the indicator random variable, equal to 1 if and only if there is a multiedge between vertices u and v. We have:

$$\mathbb{E}[X_{u,v}] \leqslant 2\binom{d_v}{2}\binom{d_v}{2}\frac{1}{n\mathbb{E}[D]-1}\frac{1}{n\mathbb{E}[D]-1-2} \leqslant \frac{d_u^2 d_v^2}{2(n\mathbb{E}[D]-2)^2}$$

Let $X = \sum_{u,v\in V} X_{u,v}$ the expected number of multiedges. We have

$$\mathbb{E}[X] \leqslant \frac{\mathbb{E}[D^2]^2}{2(\mathbb{E}[D]-\frac{2}{n})^2} \leqslant \frac{\mathbb{E}[D^2]^2}{2(1-\frac{2}{n})^2} = O(1)$$