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Solutions to Handout 8

Complex Network Models

Solution to Exercise 1

We look at the probability that the target vertex t has a single neighboring vertex of weight at most 2. With constant probability, t has a weight of at most 2. The probabilty that it is connected to another vertex v is $\Theta(1)\int_1^{n/2} w^{-\tau}\frac{w}{n}dw = \Theta(1/n)$. The probability that it connects to exactly one vertex, and that this vertex has weight at most 2 is

$$\sum_{v\neq t}\mathbb{P}\left[t\sim v, w_v \leqslant 2, \cap_{w\neq t,v} t \nsim w\right] = \sum_{v\neq t}\mathbb{P}[w_v\leqslant 2]\mathbb{P}[t\sim v \mid w_v \leqslant 2]\mathbb{P}\left[\cap_{w\neq t,v} t\nsim w\right]$$
$$= (n-1)\Theta(1)\Theta\left(\frac{1}{n}\right)\left(1-\Theta\left(\frac{1}{n}\right)\right)^{n-2} = \Theta(1)$$

We also know that in expectation, $\Theta(n)$ of the remaining vertices are of weight at most 2. As this number is a sum of Bernoulli independent random variables, we get that it is concentrated around its expectation. Hence, with constant probability, at least $\Theta(n)$ vertices have of weight of at most 2, and t hides before one of them. We need to visit in expectation at least half of them, whatever our algorithm does. With constant probability,[1] the expected running time is $\Theta(n)$, which means the expected running time is $\Theta(n)$.

Solution to Exercise 2

Since we are ignoring the cutoff, the weight of the vertex v does not matter. The probability that two fixed neighbours u and u' are connected is:

$$\begin{aligned}
\mathbb{P}[u\sim u' \mid v\sim u, v\sim u'] &= \Theta(1)\int_1^\infty\int_1^\infty x^{1-\tau}y^{1-\tau}\min\Big(\frac{xy}{n},1\Big)\,dx\,dy \\
&= \Theta(1)\int_1^n x^{1-\tau}\left[\int_1^{n/x} y^{1-\tau}\frac{xy}{n}dx\,dy + \int_{n/x}^\infty y^{1-\tau}dx\,dy\right] + \Theta(n^{2-\tau}) \\
&= \Theta(1)\left[\left(\int_1^n n^{2-\tau}/x\right) + n^{2-\tau}\right] = \Theta(n^{2-\tau}\log n),
\end{aligned}$$

where we split the above integrals at the values where the connection probability becomes constant. The probability we calculated is exactly the upper bound for the clustering coefficient (both for a fixed vertex and for the average local clustering coefficient of G).

[1] We consider the event that there are linearly many vertices of weight at most 2. This actually happens even with high probability, but we don't really care for the sake of proving large expectation.