Complex Network Models

Solution to Exercise 1

Let s and t be two random vertices. Let T be the time until we reach t starting from s following the greedy routing algorithm. We first compute the expected initial distance between s and t. We have[1] $\mathbb{E}[d_1(s,t)] = \Theta(dn^{1/d})$. As $T \leqslant d_1(s,t)$, we also have $\mathbb{E}[T] \leqslant= \Theta\left(n^{1/d}\right)$.

We define $\Delta \coloneqq p^{-\frac{1}{d+1}}$. We have $\Delta^{d+1}p = 1$. Let T' be the time until we reach a vertex at distance at most Δ from t. While this is not the case, we have for every new discovered vertex, a probability of $1-(1-p)^{\Theta(\Delta^d)} = \Theta(p\Delta^d) = \Theta(1/\Delta)$ to jump to one of the $\Theta(\Delta^d)$ vertices at distance at most Δ from t. T' is upperbounded by a geometric law of success probability $\Theta(1/\Delta)$, so $\mathbb{E}[T'] \leqslant \Theta(\Delta)$. Once we get at distance at most Δ from t, we can reach our target in at most Δ additional steps, so $T \leqslant T' + \Delta$, and $\mathbb{E}[T] \leqslant \Theta(\Delta) = \Theta\left(p^{-\frac{1}{d+1}}\right)$.

This concludes the proof.

Solution to Exercise 2

The idea is that if in k steps, we have a constant probability not to do a jump of length at least r, then we will not be able to travel a distance kr or more with constant probability.

The probability to make a jump of distance u in one step is

$$1-\left(1-\frac{1}{u^{d+\varepsilon}}\right)^{\Theta(u^{d-1})} \leqslant \frac{\Theta(u^{d-1})}{u^{d+\varepsilon}} = \Theta\left(\frac{1}{u^{1+\varepsilon}}\right)$$

the constants not depending on u. By union bound, the probability to make a jump of distance at least $r(n)$ is

$$\Theta\left(\sum_{u=r(n)} \frac{1}{u^{1+\varepsilon}}\right) = \Theta\left(\frac{1}{r(n)^{\varepsilon}}\right)$$

The probability that it doesn't happen in any of the first $k(n)$ steps is, by union bound, at most $\Theta\left(\frac{k(n)}{r(n)^{\varepsilon}}\right)$. If our initial distance is $l(n)$, and we fix $k(n)r(n) = l(n)$, then with probability $\Theta\left(\frac{l(n)}{r(n)^{1+\varepsilon}}\right)$, we won't be done after $\frac{l(n)}{r(n)}$ steps.

With constant probability, the initial distance between s and t is $l(n) = \Theta\left(n^{1/d}\right)$. We fix $r(n) = Cn^{\frac{1}{d(1+\varepsilon)}}$, with a constant small enough for the probability not to be done after $\frac{l(n)}{r(n)} = \Theta\left(n^{\frac{1}{d}-\frac{1}{d(1+\varepsilon)}}\right)$ steps to be smaller that $1/2$. With constant probability, we need a polynomial number of steps, so the expected number of steps is polynomial.

[1] if we have a torus topology, $\Theta(n^{1/d})$ else