

Ecole polytechnique fédérale de Zurich Politecnico federale di Zurigo Federal Institute of Technology at Zurich

Institute of Theoretical Computer Science Peter Widmayer Matúš Mihalák Andreas Bärtschi

Algorithmic Game Theory HS 2013

Exercise sheet 3

EXERCISE 3.1:

We want to briefly review the part of the lecture where we talked about how one can find mixed Nash equilibria. Consider a simple 2-player game:

P_2	q	1-q	expected
P_1	L	R	payoff
p T	(5,0)	(2,1)	$5q + 2 \cdot (1-q)$
(1-p) B	(1,1)	(3,0)	$1 \cdot q + 3 \cdot (1 - q)$

The following important considerations help us to find a Nash Equilibrium:

- There is no pure Nash equilibrium: in every pure strategy profile, one of the players has an incentive to switch to the clockwise next strategy profile. For the same reason, none of the players will have a mixed strategy with support size 1. Hence we can assign probabilities p, 1 p and q, 1 q to the players' strategies with $p, q \in (0, 1)$.
- Any strategy in the support of the strategy mix should be a best-response: Player 2 plays the mixed strategy $\vec{q} = (q, (1 - q))$ such that the expected payoffs for Player 1 with T and B are equal.

$$(I) \quad (A \cdot \vec{q})_1 = 5q + 2(1 - q) = 2 + 3q$$

(II)
$$(A \cdot \vec{q})_2 = q + 3(1 - q) = 3 - 2q$$

$$\Rightarrow (I) \stackrel{!}{=} (II) \Leftrightarrow 2 + 3q = 3 - 2q \Rightarrow q = \frac{1}{5}$$

Find the probability p to get a mixed NE and calculate the expected payoffs of players 1 and 2.

EXERCISE 3.2:

a) Find all pure Nash Equilibria and all mixed Nash Equilibria of the strategic game given by the following matrix (for an entry (i, j), i is the payoff of Rose and j is the payoff of Colin.):

$$\begin{array}{c|c} & & & & & \\ & & a & b & c \\ \hline \text{Rose} & A & \hline (1,2) & (2,5) & (4,4) \\ B & \hline (7,4) & (3,5) & (0,6) \end{array}$$

b) Make an example of an infinite game that has no Nash equilibrium. *Hint:* You can either take a finite set of players with infinite sets of strategies, or an infinite set of players with a finite set of strategies each.

EXERCISE 3.3:

A passenger uses a public transport, but (selfishly) he is not sure whether buying the ticket all the time is the best strategy for his finances. The passenger is thus weighing two possible strategies: to buy a ticket, or not to buy a ticket. The inspector of the public transport company has also two strategies: to inspect or not.

The ticket costs 7 CHF. Every inspection incurs a cost of 2 CHF to the inspector. If the inspector catches someone without a ticket, on top of the 2 CHF it costs the inspector an additional 1 CHF of administrative costs. The passenger without a ticket then has to pay the regular fare fee and a fine of 100 CHF. Of the total of 107 CHF, 100 CHF go to the company who runs the public transport, and 7 CHF go to the inspector, as part of his salary.

- a) Model the game as a matrix game.
- b) Compute a (mixed) Nash equilibrium of the game.
- c) What is the expected payoff to the passenger and the inspector in your equilibrium?
- d) What happens if the company decides to increase the fine from 100 to 200 CHF?

EXERCISE 3.4:

Consider a strategic game G with n players. Recall that an elimination of weakly/strictly dominated strategies is an iterative process in which we remove strategies from the game that are weakly/strictly dominated. This process results in a (possibly much smaller) sub-game G'. If the process ends in a game G' with exactly one strategy for every player, we say that the corresponding strategy profile is a solution of the game G. Prove or disprove the following assertions:

- a) Suppose that in the aforementioned process we remove only strictly dominated strategies. If a strategy profile s is a Nash equilibrium in G, then s exists in G' and is a Nash equilibrium of the game G'.
- b) If only strictly dominated strategies are removed, and the process results in a solution, then this solution is the unique Nash equilibrium of G.
- c) If also weakly dominated strategies are removed, and the process results in a solution, then this is a Nash equilibrium of G. Moreover, it is the unique Nash equilibrium of G.

EXERCISE 3.5:

Consider the following game: n pirates p_1 to p_n stole 100 gold pieces and wish to divide them. It is their custom to make such divisions in the following manner: The fiercest pirate p_n makes a proposal about the division, and everybody votes on it, including the proposer p_n . If 50 percent or more are in favour, the gold is divided according to his proposal. Otherwise the proposer is thrown overboard, and the procedure is repeated recursively with the next fiercest pirate p_{n-1} .

All the pirates enjoy throwing one of their fellows overboard, but if given the choice between throwing someone overboard and receiving some gold they prefer the gold. In particular this means that if a pirate receives the same amount of gold in two proposals P and Q, but only in Q some other pirate is thrown over board, he prefers Q. Above all, a pirate dislikes being thrown overboard himself. All pirates are rational and know that the other pirates are also rational. Moreover, no two pirates are equally fierce, i.e. there is a strict total order on the pirates p_1 to p_n in increasing fierceness (known to every pirate). A gold piece is indivisible, and arrangements to share pieces are not permitted.

What proposal should the fiercest pirate p_n make to get the most gold in each of the following cases? Justify your answers.

a) n = 2 b) n = 3 c) n = 10