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Algorithmic Game Theory HS 2013

Exercise sheet 4

EXERCISE 4.1:
Consider a *singleton* congestion game of n players, in which every player's strategy is to choose exactly one resource from the set R of m resources. Show that every better-response dynamics finds a pure Nash equilibrium in time that is polynomial in n and m. For this purpose, modify the cost funtion c_r of every resource $r \in R$ to express its *rank* among all values of $c_r(n_r)$, $r \in R$, $n_r \in \{1, 2, \ldots, n\}$. The rank $\overline{c}_r(n_r)$ of a value $c_r(n_r)$ is its position in a sorted list of all the values $c_r(n_r)$, where the same values get the same rank, i.e.

$$\overline{c}_r(i) := |\{c_p(j) \mid \exists p \in R \text{ and } \exists j \in \{1, \ldots, n\} \text{ such that } c_p(j) \leq c_r(i)\}| .$$

Show that the game "behaves" the same in the modified values and use it to bound the number of steps of any better-response dynamics.

EXERCISE 4.2:
A transition graph $T_G = (V, E)$ of a finite strategic game G is a directed graph in which every vertex $v_s \in V$ corresponds to a strategy profile $s \in S$ and there is an edge from v_s to v_t if $t = (s'_i, s_{-i})$ and s'_i is a better response to s for some player i. (Recall that $s'_i \in S_i$ is a better-response of player i to the strategy profile $s = (s'_i, s_{-i}) = (s_1, \cdots, s_i, \cdots, s_n)$ if $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$.
Show that every finite strategic game whose transition graph is acyclic, is an ordinal potential game by giving an appropriate ordinal potential function and proving that it is indeed a potential function. (Recall that a function $\Phi : S \to \mathbb{R}$ is an ordinal potential function if it holds that whenever $\Delta u_i < 0$ then $\Delta\Phi < 0$, i.e., if player i changes his strategy and improves his payoff then the function Φ changes in the same "direction".)

EXERCISE 4.3:
A 2-player zero-sum game is a strategic game in which $u_1(s) = -u_2(s)$ for all strategy profiles s. Let G be a 2-player zero-sum game where the first player has n strategies while the second has m, i.e. $|S_1| = n$ and $|S_2| = m$. Assume there are two pure Nash equilibria in G constituted by the two respective strategy profiles $s = (s_1, s_2)$ and $t = (t_1, t_2)$.

a) Prove that the payoffs of the two Nash equilibria are the same, i.e. show that $u_i(s) = u_i(t)$ for $i \in \{1, 2\}$.

b) Show that (s_1, t_2) and (t_1, s_2) are Nash equilibria of G as well.

EXERCISE 4.4:
In the lecture we have seen that a better-response dynamics in congestion games will lead to a pure-strategy Nash equilibrium. What about strategic games that do have a pure-strategy Nash equilibrium?
Find a strategic game with a pure Nash equilibrium s and a strategy profile s' such that there is a best-response dynamics leading from s' to s, while at the same time there is a cyclic (and non-empty) best-response dynamics leading from s' to s'.

EXERCISE 4.5:

Will not be fully discussed in the exercise classes.

Consider the two-player strategic game given by the following matrix in which the payoff to the players Rose and Colin are the first and second entry in each tuple, respectively. Both players want to maximise their payoff.

Colin

Rose		c_1	c_2	c_3	c_4
	r_1	$(5, 10)$	$(10, 5)$	$(2, 5)$	$(7, 12)$
	r_2	$(10, 9)$	$(5, 6)$	$(7, 3)$	$(8, 7)$
	r_3	$(8, 4)$	$(8, 9)$	$(11, 1)$	$(6, 3)$
	r_4	$(9, 1)$	$(3, 8)$	$(6, 7)$	$(7, 3)$

a) Find all strictly and weakly dominated strategies of the above game.

b) Try to solve the above game by an iterative elimination of strictly dominated strategies. Give the order in which you removed strategies from the game. What is the remaining subgame?

c) Find all (mixed and pure) Nash equilibria of the game.