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Algorithmic Game Theory HS 2013

Exercise sheet 5

EXERCISE 5.1:
In the last exercise sheet we considered singleton congestion games and showed that a Nash equilibrium can always be reached in a polynomial number of steps. In this exercise we want to look at a generalization to congestion games where we identify a subset $R_i \subseteq R$ of the resources with each player i. She then chooses an arbitrary subset of size exactly k_i of R_i as her strategy, i.e. $S_i = \{s | s \subseteq R_i \text{ and } |s| = k_i\}$. There are n players, m resources, and the cost function of each resource takes positive natural numbers.

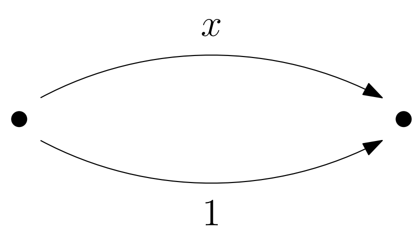
a) Show that there exist games of this type for which a *better*-response dynamics may need an exponential number of steps to reach a Nash equilibrium.

   Hint. Consider a game with only one player.

b) Explain where the proof with the rank function $\bar{c}_r(i)$ from the last exercise sheet breaks down. Recall the definition of $\bar{c}_r(i)$:

$$\bar{c}_r(i) = |\{c_p(j) | \exists p \in R \text{ and } \exists j \in \{1, \ldots, n\} \text{ s.t. } c_p(j) \leq c_r(i)\}|.$$

EXERCISE 5.2:
The so called Pigou example (depicted below) is the famous simple example that shows that the price of anarchy in routing games is at least 4/3. We achieve this by setting the lower link to have a constant latency function 1 and the upper link to have a linear latency x. Consider a situation where no constant function can be used as a latency function, but only the identity function $l_e^{\text{ID}}(x) = x$ and some continuous non-decreasing function l_e, for which we know that $l_e(0) = 1$. Given some arbitrary such function l_e and the identity function l_e^{ID}, can you modify Pigou's example (by changing the topology and using only the two types of functions) and come arbitrarily close to the ratio 4/3?



EXERCISE 5.3:
Consider the routing game from the lecture and the proof showing that the price of anarchy is at most 4/3. Prove the following, not-presented part of the proof.

Given a Nash flow f (i.e., a flow that is Nash equilibrium) and an arbitrary flow f' (feasible for the problem), show that

$$\sum_{e \in E} l_e(f_e) \cdot f_e \leq \sum_{e \in E} l_e(f_e) \cdot f'_e.$$

EXERCISE 5.4:
Recall the *Load Balancing Game* presented in the lecture, where we have m machines $M_1, \ldots, M_m$ and n jobs with execution times $w_1, \ldots, w_n$, respectively. Our purpose is to provide a schedule for the jobs that minimizes the time at which every job is finished (the so called MAKESPAN).

a) We want to prove that the game is a potential game. Recall the definition of

$$\begin{aligned} \Phi \colon S &\longrightarrow X \\ s &\longmapsto \Phi(s) := (\alpha_1, \alpha_2, \ldots, \alpha_m), \end{aligned}$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ is the ordered sequence of $\{l_1(s), \ldots, l_m(s)\}$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m$. Then X is totally ordered (by the lexicographical ordering). Complete the proof by showing that Φ is a potential function, i.e. prove that:

$$s \text{ is not a NE } \Rightarrow \forall \text{ better-responses } s'_i \text{ we have } \Phi(s') < \Phi(s),\ s' = (s'_i, s_{-i}).$$

Hint: Look at the load vector of s and a player i who wants to switch his job w_i from x to y: $(\ldots, x + w_i, \ldots, y, \ldots)$. Argue that then s' comes lexicographically before s.

b) Show that even in the *Generalized Load Balancing Game*, where every machine $M_1, \ldots, M_m$ has a different speed, the load balancing game is still a potential game. The speed of a machine M_j is a constant factor s_j such that every job of weight w_i on machine j takes $\frac{w_i}{s_j}$ time.