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# **Algorithmic Game Theory** HS 2013

Exercise sheet 6

## EXERCISE 6.1:

Consider a *Load Balancing Game* from the lecture, where we have $m$ machines $M_1,\ldots,M_m$ and $n$ jobs with execution times $w_1, \ldots, w_n$, respectively. The jobs are the players and they are interested to be assigned at a machine which finishes earliest possible. Consider the following best-response dynamics. Starting in time $t = 1$ from an arbitrary strategy profile $s = (s_1, \ldots, s_n)$ (where $s_i$ is the machine to which job $i$ is assigned), perform in every step $t$ the following: If the current strategy profile $s$ is a Nash equilibrium, stop the process; Else, among all players that can improve their cost by unilaterally chaning its strategy in $s$, identify the player $i$ with highest processing time $w_i$; Let this player play a best-response $s'_i$ to obtain a new current strategy profile $s' = (s_1, \ldots, s'_i, \ldots, s_n)$.

Show that the described best-response dynamics always finds a pure-strategy Nash equilibrium where each player plays at most *once* a best-response. (And thus, the best-response dynamics finds a Nash equilibrium in at most $n$ steps.)

## EXERCISE 6.2:

Recall from the lecture the *Local Connection Game*, where $n$ players are the vertices $V = \{1, 2, \ldots, n\}$ (of an initially "empty" graph), and they can buy *adjacent* edges, each at a fixed price $\alpha > 0$. Their goal is to create "cheaply" a "good network", i.e., not to pay much and end up with a network that provides short distances to other nodes. Formally, each player $i$ wants to minimize

$$cost_i(s) = \alpha \cdot |E_i| + \sum_{j=1}^{n} \mathbf{dist}(i, j),$$

where $E_i$ are the edges bought by player $i$ and $\mathbf{dist}(i, j)$ is the distance between $i$ and $j$ in the resulting graph $G = (V, \bigcup_{i=1}^{n} E_i)$.
We have seen that, if $\alpha > 2$, the star network is a Nash equilibrium for this game.

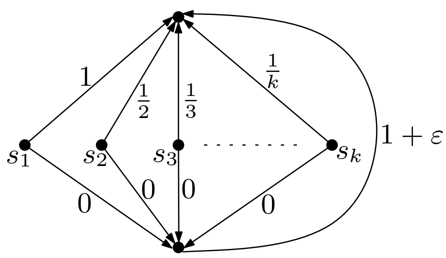
a) Construct a Nash equilibrium that is not a star for $\alpha > 2$. You can choose your $\alpha > 2$ as you wish. Can you construct a Nash equilibrium that is not a tree?

b) Show that when $\alpha > n^2$, all Nash equilibria of the local connection game are trees and the *Price of Anarchy* is bounded by a constant.

**EXERCISE 6.3:**
In this exercise we consider the *global* connection game. Recall that in this game there is a graph $G = (V, E)$ with costs on the edges and a set of $k$ players where every player $i$ wants to connect a node $s_i$ with a node $t_i$ using a directed path in $G$. Thus, a strategy of player $i$ is a path from $s_i$ to $t_i$. Every player pays his fair deal for every edge of the chosen path, i.e., if an edge $e \in E$ of cost $c_e$ is chosen by $k_e$ players (in their chosen paths) in a strategy profile, then every such player pays $c_e/k_e$ as a contribution towards the total cost of the edge. The total cost of player $i$ is then the sum of all his contributions.

a) Prove that in any such game with $k$ players, the price of anarchy is at most $k$.

b) At the lecture we have seen that for *directed graphs*, the price of stability can be as high as $H_k/(1+\varepsilon)$ (see also the figure below). This example relies on the carefully chosen edge-costs. Analyse the price of stability for unweighted networks, i.e., for directed graphs where each edge has unit cost. What is the upper bound on the price of stability (in any such unweighted game)? How bad can the price of stability be (in some unweighted game)?




c) Analyse the price of stability for *undirected graphs* with two players (i.e., $k = 2$), where $t_1 = t_2$, i.e., the players want to connect their sources $s_i$, $i = 1, 2$ with the common target $t = t_1 = t_2$. In particular,

- show that the price of stability is at most $4/3$ (and thus strictly smaller than $H_2 = 3/2$). It is enough to consider a Nash equilibrium $N$ that minimizes the Rosenthal's potential function $\Phi$, and thus it holds that $\Phi(N) \le \Phi(\mathrm{OPT})$, where OPT is the optimum strategy profile (minimizing the total cost). You may want to use the fact that in Nash equilibrium, no player wants to deviate from the chosen path. In particular, player 1 does not want to "go from $s_1$ to $s_2$ using the graph formed by an optimum solution, and then using the strategy of player 2 to get to the common vertex $t$".
- Construct an example with as high price of stability as possible.