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Algorithmic Game Theory HS 2013

Exercise sheet 7

EXERCISE 7.1:
Recall the following two definitions (where L denotes the set of all the possible preferences of the players over the set of alternatives A).

i) A social choice function $f\colon L^n \to L$ is *truthful* if it cannot be *strategically manipulated.*
A player i can strategically manipulate a social choice function $f\colon L^n \to L$ if there is a strategy profile $s := (\succ_1, \dots, \succ_n) \in L^n$ and some alternate strategy $\succ_i' \in L$, such that player i already preferred a' to a in his original strategy $\succ_i$, but by changing some of his preferences he achieves that a' is getting elected, i.e. $a' \succ_i a$, where $a = f(s)$ and $a' = f(\succ_i', s_{-i})$.

ii) A social choice function f is *monotone* if the strategic change from $\succ_i$ to $\succ_i'$ of a player i, which is causing the outcome of f to change from a to a', implies that this player changed his preference from $a \succ_i a'$ to $a' \succ_i' a$, i.e.

$$\forall \text{ preference profiles } s := (\succ_1 \dots, \succ_i, \dots, \succ_n), \forall \succ_i' \text{ s.t. } f(s) = a, f(\succ_i', s_{-i}) = a' \implies a \succ_i a', a' \succ_i' a.$$

Prove that a social choice function is truthful if and only if it is monotone.

EXERCISE 7.2:
Consider the following questions regarding social welfare and social choice functions.

a) If F is a dictatorship, how many dictators can it have?

b) Consider elections in which there are only two candidates a and b, i.e. $|A| = 2$. Thus the preference of every player i is either $a \succ_i b$ or $b \succ_i a$. The *majority* vote between two candidates chooses the candidate that is preferred by the majority of the players. In case of a tie, a is chosen. Show that the majority vote, considered as a social choice function, is truthful, i.e. no player can strategically manipulate this voting system.

c) Can you devise for the case of b) a social welfare function F which satisfies unanimity, consistency, and which is not a dictatorship?

d) Can you devise a truthful social choice function f for the case when $|A| \geq 3$, which is not onto (that is, at least a candidate can never be elected)?

e) How many different social welfare functions that satisfy unanimity and consistency are there for the setting with n voters and $|A| \geq 3$?

f) What can you say in e) if $|A| = 2$?

g) Consider the following social choice function on $|A| \geq 3$: Take a pair of candidates, check which candidate is preferred by the majority and discard the other candidate. Iterate until only one candidate is left. Is this election scheme truthful?

EXERCISE 7.3:
In the proof of Arrow's theorem, we use the following auxiliary lemma, which generalizes the consistency property (also known as the independence of irrelevant alternatives) of a social welfare function.

Lemma (Pairwise Neutrality): Every social welfare function F that satisfies unanimity and consistency also satisfies pairwise neutrality, i.e.

$$\forall a, b, c, d \in A, \forall(\succ_1, \ldots, \succ_n) \in L^n, \forall(\succ'_1, \ldots, \succ'_n) \in L^n : \text{ If } \forall i(a \succ_i b \Leftrightarrow c \succ'_i b) \text{ then } a \succ b \Leftrightarrow c \succ' b.$$

Prove the lemma for the special case where c, d are different from a, b.
Hint: Try to "merge" preferences.

EXERCISE 7.4:
Finish the proof of the Gibbard-Satterthwaite theorem from the lecture, i.e., prove that the social welfare function F defined from the social choice function $f : L^n \to A$ (which is onto, and satisfies unanimity, consistency, and is not a dictatorship), is not a dictatorship.
Recall that F has been defined by setting $a \succ b$ iff $f(\succ_1^{\{a,b\}}, \succ_2^{\{a,b\}}, \ldots, \succ_n^{\{a,b\}}) = a$.