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Algorithmic Game Theory HS 2013

Exercise sheet 8

EXERCISE 8.1:

Let an instance of the *House Allocation* problem with n players be given. Consider the following algorithm that is equivalent to the *Top Trading Cycle algorithm* (TTCA) presented in the lecture. It operates on the following (complete) directed graph, where every player is represented by a vertex. If house j is player i's kth choice, we add a directed edge (i, j) of color k. The algorithm works as follows: in every iteration $i = 1, \ldots, n$ every player considers her best option (i.e., the outgoing edge pointing to her best rated house of the available ones) in the current graph. This operation possibly induces node-disjoint directed cycles and loops. Let N_i be the set of players that form these cycles in iteration i. The algorithm reassigns the houses to the players in N_i consistently according to their preferences. Before starting the next iteration it removes the nodes corresponding to N_i (and their incident edges) from the graph and it increases i.

a) Apply the TTCA to the following instance with players a, b, c, d.

$$\begin{aligned} a &: \; b \succ c \succ a \succ d \\ b &: \; c \succ a \succ b \succ d \\ c &: \; d \succ a \succ c \succ b \\ d &: \; d \succ c \succ a \succ b \end{aligned}$$

b) Prove that the outcome of the TTCA is in the core of the House Allocation problem, that is, there is no blocking coalition S among the players for the allocation produced by TTCA.

c) Consider the following modified version of the TTCA (which is the algorithm given in the AGT book). At every iteration i we look only at the loops and cycles that have color i and again we set N_i as the set of players in that loops and cycles. We reallocate the houses among N_i in the natural way, following the preferences induced by the directed edge in every such cycle (loop). After the corresponding reallocations have been done, we remove all the edges of color i and all players N_i and we iterate.

   Does the output of this algorithm belong to the core of the House Allocation problem? Provide an argument or a counterexample.

EXERCISE 8.2:

Show that the male-proposal algorithm for stable marriages is not truthful for women.

EXERCISE 8.3:

Let the unit interval $[0,1]$ be the set of alternatives of an election. There are n voters, each voter i having a single most preferred alternative $t_i \in [0,1]$. The player i's preferences of other alternatives are decreasing linearly with the increasing distance from t_i, i.e., player i prefers alternative $x \in [0,1]$ to alternative $y \in [0,1]$ if (and only if) $|x - t_i| < |y - t_i|$. Player i is indifferent between x and y if they are at the same distance from t_i. Thus, the preferences of player i are determined by a single value t_i. The voters vote by casting a number from the interval $[0,1]$ (which is handled by the election as the most preferred alternative). The outcome of the election (as a result of a social choice function) is a single number in the interval $[0,1]$.

a) Show that the social choice function that returns the average of the casted numbers (the claimed most preferred alternatives) is not truthful.

b) Devise a truthful social choice function f, which is *onto* and *anonymous*. A social function f is anonymous if $f(s_1, \ldots, s_n) = f(s_{\pi(1)}, \ldots, s_{\pi(n)})$, for every permutation π of the set $\{1, \ldots, n\}$. In other words, f is independent of the name of the players (and thus cannot be a dictatorship).
Remark 1: a function can not choose a random number. That would not be a function.
Remark 2: Gibbard-Satterthwaite's theorem does not apply here, since here the players do not have arbitrary preferences about the alternatives.