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Algorithmic Game Theory HS 2013

Exercise sheet 11

In the *Generalized Second-Price (GSP) Auction*, n advertisement slots with *click-through rates* $r_1 \geq r_2 \geq \ldots \geq r_n \geq 0$, are auctioned off to n buyers with per-click valuations $v_1 \geq v_2 \geq \ldots \geq v_n > 0$. In GSP, each buyer i casts a bid b_i, the mechanism sorts the bids in a decreasing order, assigns the k-th highest bidder the corresponding slot k, $k = 1, \ldots, n$, and charges it the price-per-click equal to the bid b_{k+1} (with the convention that $b_{n+1} := 0$). Let $\pi(j)$ denote the player that is assigned to slot j, $j = 1, 2, \ldots, n$. (Observe that π is a permutation of $(1, 2, \ldots, n)$.) Then, the valuation of player $\pi(j)$ of the slot j is $v_{\pi(j)} \cdot r_j$, and the payment of the player is $b_{\pi(j+1)} \cdot r_j$. Thus, the player's *payoff* is $v_{\pi(j)} \cdot r_j - b_{\pi(j+1)} \cdot r_j$.

EXERCISE 11.1:

Consider the GSP auction as a strategic game, where the strategies of players are the bids b_i, and the payoffs are given by the result of the GSP auction. The *social welfare* of a strategic profile $b = (b_1, \ldots, b_n)$ is the total valuation of the slots assigned to players, i.e., $\mathrm{SW}(b) := \sum_{j=1}^{n} v_{\pi(j)} \cdot r_j$. An assignment of slots to bidders that maximizes this sum is called a *social optimum*, and denoted by OPT. We are interested in price of anarchy (PoA), which is now defined as the ratio

$$\frac{\mathrm{SW(OPT)}}{\mathrm{SW(worst\ NE)}} = \frac{\sum_{j=1}^{n} v_j \cdot r_j}{\sum_{j=1}^{n} v_{\pi^*(j)} \cdot r_j},$$

where $\pi^*(j)$, $j = 1, \ldots, n$ is the assignment of buyers to slots in a pure-strategy Nash equilibrium of largest social welfare.

a) Show that an assignment where $\pi(j) = j$ is a social optimum, having social welfare $\sum_{j=1}^{n} v_j \cdot r_j$.

b) Show that PoA can be arbitrary bad, i.e., show that for any $\alpha > 1$ there exists a setting in which the PoA is larger than α.
(Hint: You may consider a Vickrey auction translated into GSP.)

c) Consider the *restriction* of GSP game in which every player i can only bid $b_i \leq v_i$ (i.e., the set of strategies S_i is equal to $\{x \,:\, x \leq v_i\}$.

  i) Consider a pure Nash equilibrium, and let $\pi(j)$ be the buyer assigned to slot j in this Nash equilibrium. Prove that for every j and j', the following holds:

$$v_{\pi(j')} \cdot r_{j'} + v_{\pi(j)} \cdot r_j \geq v_{\pi(j')} \cdot r_j,$$

  or, equivalently,

$$\frac{r_{j'}}{r_j} + \frac{v_{\pi(j)}}{v_{\pi(j')}} \geq 1.$$

  (Hint: Use the fact that in NE no player wants to change its strategy to obtain a different slot.)

  ii) Using the above inequality, prove that in any restricted GSP game, the price of anarchy is at most two.