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Algorithmic Game Theory HS 2013

Exercise sheet 12

EXERCISE 12.1:
Let G be a countably infinite connected graph with vertex set V and finite maximum degree $\Delta(G)$. The vertices are called the players, and they adapt to one of the strategies A, B. We fix a threshold $q \in [0,1]$ and an initial set of players $S \subset V$ with behaviour B. For a player v, we denote with d_v^A (respectively d_v^B) the number of its neighbours with behaviour A (respectively B). In each step t, v chooses strategy B if and only if $d_v^B \geq q \cdot (d_v^A + d_v^B)$. v is called *converted* by S, if after some time k, v sticks to B for all $t \geq k$. S is called *contagious* if it converts every node in V.

a) Find a non-trivial threshold $q > 0$ together with a finite contagious initial set S.

b) Let $q = \frac{1}{\Delta(G)}$ and $S = \{x, y\}$. Show that if x and y have an odd distance, then S is contagious.

c) Let $q = \frac{1}{\Delta(G)}$ and $S = \{x, y\}$. Show that if S is not contagious, then G is a bipartite graph.

EXERCISE 12.2:
In this exercise we want to prove the following fact used in the lecture: A weighted sum of submodular functions is a submodular function itself. Recall that a *submodular function* is a set function $f\colon 2^V \to \mathbb{R}$ which satisfies one of the following equivalent definitions:

1. For every $X, Y \subseteq V$ with $X \subseteq Y$ and every $v \in V \setminus Y$ we have that $f(X \cup \{v\}) - f(X) \geq f(Y \cup \{v\}) - f(Y)$.

2. For every $S, T \subseteq V$ we have that $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$.

Show that for given submodular functions $f_1, \ldots, f_r$ and constants $c_1, \ldots, c_r \geq 0$, the linear combination $f = c_1 \cdot f_1 + \ldots c_r \cdot f_r$ is submodular.

EXERCISE 12.3:
Let $X_1, \ldots X_r$ be arbitrary finite sets and let $\{1, 2, \ldots, r\}$ be the set of their indices. Then we can define the so-called "size-of-union" function f:

$$\forall I \subseteq \{1, 2, \ldots, r\},\ f(I) = \left| \bigcup_{i \in I} X_i \right|.$$

Show that f is submodular.