1 Union bound

A very important tool in discrete probability theory is the so called union bound. This bound will occur numerous times throughout the course. We state it without giving a proof here.

Theorem 1. *Let* $(\Omega, \Pr)$ *be a discrete probability space and let* $A_1, A_2, \ldots, A_n \subseteq \Omega$ *be events. Then we have*

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \Pr[A_i].$$

2 Landau Symbols / $\mathcal{O}$-Notation

Throughout the course we will use the so called Landau symbols to describe asymptotic behavior of functions. For two functions, $f, g : \mathbb{N} \to \mathbb{R}$ we write

$$\begin{aligned}
f = \mathcal{O}(g) \qquad & \text{if } 0 \leq \limsup_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| < \infty, \\
f = o(g) \text{ or } f \ll g \quad & \text{if } \lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = 0, \\
f = \Omega(g) \qquad & \text{if } 0 < \liminf_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| \leq \infty, \\
f = \omega(g) \text{ or } f \gg g \quad & \text{if } \lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = \infty, \\
f = \Theta(g) \qquad & \text{if } f = \Omega(g) \text{ and } f = \mathcal{O}(n).
\end{aligned}$$

An alternative equivalent definition for $\mathcal{O}, \Omega$ and Θ is

$$\begin{aligned}
f = \mathcal{O}(g) \quad & \text{if } \exists C > 0, n_0 : \forall n \geq n_0 : |f(n)| \leq C|g(n)|, \\
f = \Omega(g) \quad & \text{if } \exists c > 0, n_0 : \forall n \geq n_0 : |f(n)| \geq c|g(n)|, \\
f = \Theta(g) \quad & \text{if } \exists c > 0, C > 0, n_0 : \forall n \geq n_0 : c|g(n)| \leq |f(n)| \leq C|g(n)|.
\end{aligned}$$

2.1 Examples

- $1000n = \Theta(n)$,
- $n = o(n^{1+\varepsilon})$ for every $\varepsilon > 0$,
- $n^{100} = o(\log(n)^{\log(n)})$ since $\log(n)^{\log(n)} = n^{\log\log(n)}$,
- $\log(n)^\delta = o(n^\varepsilon)$ for all constants $\delta, \varepsilon > 0$ since $\log(n)^\delta = e^{\delta \log\log n}$ and $n^\varepsilon = e^{\varepsilon \log n}$,

- $e^{-\frac{\Omega(n)}{o(n)}} = e^{-\omega(1)} = o(1)$, but *not* (!!) $e^{\frac{\mathcal{O}(n)}{\mathcal{O}(n)}} = e^{\mathcal{O}(1)} = \mathcal{O}(1)$ since the first equality is *not* correct since the denominator might be of order of magnitude smaller than the numerator in which case the exponent tends to infinity! The second equality is correct though.

3 Law of Total Probability / Law of Total Expectation

A very useful tool in discrete probability are the laws of total probability and total expectation.

Theorem 2. *Let* $(\Omega, \Pr)$ *be a discrete probability space and let* $A_1, A_2, \ldots, A_n \subseteq \Omega$ *be events that form a partition of* Ω*, that is, the events are pairwise disjoint and their union is* Ω*. Then we have for every event* $\mathcal{E} \subseteq \Omega$ *that*

$$\Pr[\mathcal{E}] = \sum_{i=1}^{n} \Pr[\mathcal{E}|A_i] \Pr[A_i],$$

and for every random variable X *that*

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X|A_i] \Pr[A_i].$$

Proof. We first prove that law of total probability. Note that sicne the A_i's are pairwise disjoint, the events $\mathcal{E} \cap A_1, \ldots, \mathcal{E} \cap A_n$ are also pairwise disjoint. Hence, we have

$$\sum_{i=1}^{n} \Pr[\mathcal{E} \cap A_i] = \Pr\left[\bigcup_{i=1}^{n} \mathcal{E} \cap A_i\right]. \tag{1}$$

With this we can derive

$$\Pr[\mathcal{E}] = \sum_{i=1}^{n} \Pr[\mathcal{E}|A_i] \Pr[A_i] = \sum_{i=1}^{n} \frac{\Pr[\mathcal{E} \cap A_i]}{\Pr[A_i]} \Pr[A_i] \overset{(1)}{=} \Pr\left[\bigcup_{i=1}^{n} \mathcal{E} \cap A_i\right] = \Pr[\mathcal{E}],$$

where the last step follows from the fact that the union of the A_i's equals Ω.

It remains to prove the law of total expecation. We have

$$\begin{aligned}
\sum_{i=1}^{n} \Pr[A_i]\,\mathbb{E}[X|A_i] &= \sum_{i=1}^{n} \Pr[A_i] \sum_{x\in\Omega} x\Pr[X=x|A_i] \\
&= \sum_{i=1}^{n} \sum_{x\in\Omega} x\Pr[X=x|A_i]\Pr[A_i] \\
&= \sum_{x\in\Omega} \sum_{i=1}^{n} x\Pr[X=x|A_i]\Pr[A_i] \\
&= \sum_{x\in\Omega} x \sum_{i=1}^{n} \Pr[X=x|A_i]\Pr[A_i] \\
&= \sum_{x\in\Omega} x \sum_{i=1}^{n} \Pr[X=x \ \wedge \ A_i] \\
&= \sum_{x\in\Omega} x\Pr[X=x] = \mathbb{E}[X].
\end{aligned}$$

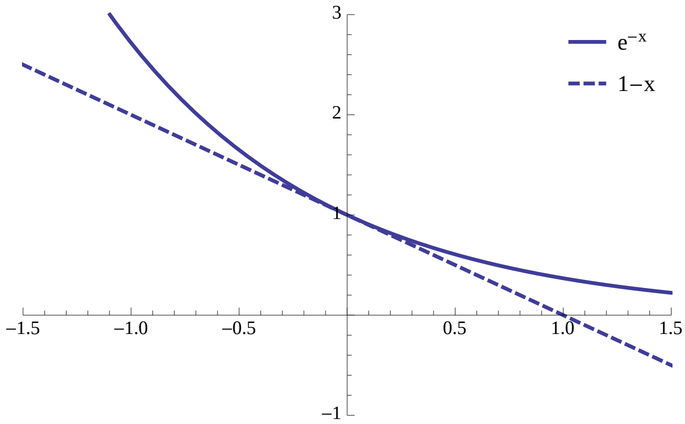
□

4 Useful Inequalities

Inequality 3. *For all $x \in \mathbb{R}$ we have*

$$1 - x \leq e^{-x}.$$

Proof. We just give an informal picture proof. □



Inequality 4. *For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k \leq n$ we have*

(i) $\binom{n}{k} \leq 2^n$,

(ii) $\binom{n}{k} \leq \frac{n^k}{k!}$, *and*

(iii) $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$.

Proof. (i) We have

$$\binom{n}{k} \leq \sum_{i=0}^{n} \binom{n}{k} = 2^n.$$

(ii) We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \leq \frac{n^k}{k!}.$$

(iii) We first $\binom{n}{k} \geq (n/k)^k$. We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \geq \left(\frac{n}{k}\right)^k,$$

where the last inequality follows from the fact that for every $k \leq n$ we have $(n-i)/(k-i) \geq n/k$. It remains to show $\binom{n}{k} \leq (ne/k)^k$. We have

$$e^k = \sum_{i=0}^{\infty} \frac{k^i}{i!} \geq \frac{k^k}{k!} = \frac{n(n-1)\cdots(n-(k-1))}{k!} \cdot \frac{k^k}{n(n-1)\cdots(n-(k-1))} \geq \binom{n}{k}\left(\frac{k}{n}\right)^k,$$

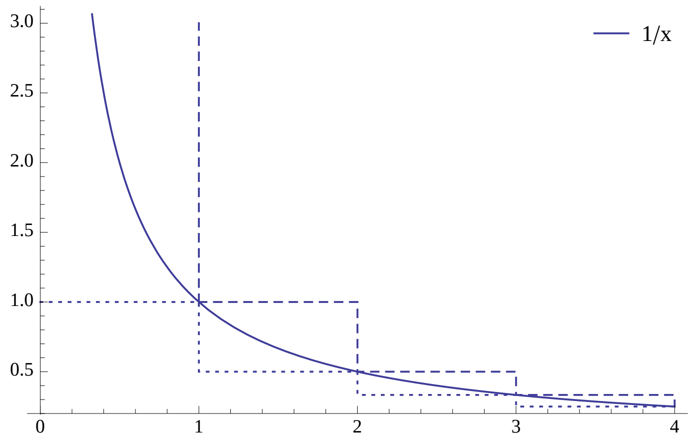
which immediately implies the claim.

□

Inequality 5. *Let* $H_n = \sum_{i=1}^{n} \frac{1}{i}$ *denote the n-th harmonic number. Then we have*

$$\ln n \leq H_n \leq \ln n + 1.$$

Proof. Recall that $\ln = \int_1^n 1/x\mathrm{d}x$. The following picture illustrates that $H_n \leq \ln(n) + 1$ and, by shifting the $1/x$-curve one unit to the left, that $\ln(n-1) \leq H_n$.



In fact, one can show that $H_n = \ln n + \gamma + \mathcal{O}(n^{-1})$ where $\gamma \approx 0.5772$ denotes the Euler-Mascheroni constant. □