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Datenstrukturen & Algorithmen — Exercise Sheet 1 — FS 16

Exercise class (Room & Day): ______________________

Submitted by: ______________________

Corrected by: ______________________

Bonus points: ____

Please use the definitions of $\mathcal{O}, \Omega$, and Θ given at the lecture for the exercises on this sheet. Namely, for a function $g : \mathbb{N} \to \mathbb{R}^+$, let

$$\mathcal{O}(g) := \{ f : \mathbb{N} \to \mathbb{R}^+ \mid \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N}, \forall n \geq n_0 : f(n) \leq cg(n) \}. \quad (1)$$

Analogously, we say that f grows asymptotically at least as much as g, if $f \in \Omega(g)$ with

$$\Omega(g) := \{ f : \mathbb{N} \to \mathbb{R}^+ \mid \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N}, \forall n \geq n_0 : f(n) \geq cg(n) \}. \quad (2)$$

A function f grows asymptotically like g when $f \in \mathcal{O}(g)$ and $f \in \Omega(g)$. We denote this by $f \in \Theta(g)$, or as $f = \Theta(g)$.

Exercise 1.1 *Examples.*

Prove the following statements by finding appropriate constants c and n_0 (exercises a), c), d)), respectively argue why such constants do not exist (exercise b)).

a) $3n^2 + 5n + 10 \in \mathcal{O}(n^2)$,

b) $n^{1/2} \notin \mathcal{O}(n^{1/3})$

c) $2^n \in \mathcal{O}(3^n)$

d) $\log_{10}(n) \in \mathcal{O}(n^{0.1})$
(Hint: Show that $\frac{\log_{10}(n)}{n^{0.1}} \to 0$ for $n \to \infty$ using L'Hôpital's rule and deduce the statement).

Exercise 1.2 *Simplifying Expressions in $\mathcal{O}$ Notation.*

Usually we simplify expressions in $\mathcal{O}$ Notation as much as possible. For example, we do not write $\mathcal{O}(3n+1)$, but $\mathcal{O}(n)$ instead. Simplify the following expressions in $\mathcal{O}$ Notation as much as possible.

a) $\mathcal{O}(2n + 14n^2)$,

b) $\mathcal{O}(\log n + 3\sqrt{n})$,

c) $\mathcal{O}(16 \log_7(n^2))$,

d) $\mathcal{O}(6^{10} + \log^5(n) + \frac{n}{10})$.

Please turn over.

Exercise 1.3 *Proofs about $\mathcal{O}$ Notation.*

Prove or disprove the following statements, where $f, g : \mathbb{N} \to \mathbb{R}^+$.

a) $f \in \mathcal{O}(g)$ if and only if $g \in \Omega(f)$.

b) If $f \in \mathcal{O}(g)$, then $f(n) \leq g(n)$ for every $n \in \mathbb{N}$.

c) There exist a function f, such that neither $f \in \mathcal{O}(n)$ nor $f \in \Omega(n)$ holds.

d) $\log_a(n) \in \Theta(\log_b(n))$ for all constants $a, b \in \mathbb{N}\backslash\{1\}$.

e) Let $f_1, f_2 \in \mathcal{O}(g)$ and $f(n) := f_1(n) + f_2(n)$. Then, $f \in \mathcal{O}(g)$.

f) Let $f_1, f_2 \in \mathcal{O}(g)$ and $f(n) := f_1(n) \cdot f_2(n)$. Then, $f \in \mathcal{O}(g)$.

Exercise 1.4 *Asymptotic Growth of Functions.*

Sort the following functions from left to right such that: if function f is to the left of g, then $f \in \mathcal{O}(g)$.

Example: the functions n^3, n^7, n^9 are already in the right order since $n^3 \in \mathcal{O}(n^7)$ and $n^7 \in \mathcal{O}(n^9)$.

$$n \log^3(n),\ \frac{n^2}{\log(n)},\ \log(2^n),\ 8^{11},\ n^n,\ n^3 + 7n,\ \sqrt{n},\ \log^5(n),\ 2^n$$

Hand-in: Thursday, 29th September 2016 before the start of the lecture at 10:00 in the entrance area of ML D28.