Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Federal Institute of Technology at Zurich

Department of Computer Science
Markus Püschel
Peter Widmayer
Thomas Tschager
Tobias Pröger

29th September 2016

Data Structures & Algorithm — Solutions to Sheet 1 — AS 16

Solution 1.1 Examples.

a) We have $5n \leq n^2$ for $n \geq 5$, and $10 \leq n^2$ for $n \geq 4$ (in particular for $n \geq 5$). We choose $n_0 = 5$ and $c = 5$ and get

$$3n^2 + 5n + 10 \leq 3n^2 + n^2 + n^2 = 5n^2 = cn^2 \text{ for all } n \geq n_0 = 5. \tag{1}$$

Consequently, $3n^2 + 5n + 10 \in \mathcal{O}(n^2)$.

b) Suppose that $n^{1/2} \in \mathcal{O}(n^{1/3})$. Then there exist constants $c > 0$ and n_0, so that $n^{1/2} \leq cn^{1/3}$ for all $n \geq n_0$. But then it would hold that $\frac{n^{1/2}}{n^{1/3}} = n^{1/2-1/3} = n^{1/6} \leq c$, which is not possible: The function $n^{1/6}$ converges for $n \to \infty$ to ∞. Hence, there is no *constant* c that leads to an upper bound for $n^{1/6}$.

c) We can choose $c = n_0 = 1$ and it holds that $2^n \leq 3^n = c \cdot 3^n$ for all $n \geq n_0 = 1$.

d) The rule of L'Hôpital states that for two differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) \to \infty$ and $g(x) \to \infty$ for $x \to \infty$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \tag{2}$$

applies if $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists.

We set $f(x) = \log_{10}(x) = \ln(x)/\ln(10)$ and $g(x) = n^{0.1}$ and get

$$\frac{f'(x)}{g'(x)} = \frac{1/(x\ln(10))}{0.1x^{-0.9}} = \frac{10x^{0.9}}{x\ln(10)} = \frac{10}{\ln(10)x^{0.1}} \xrightarrow[x\to\infty]{} 0. \tag{3}$$

Consequently, according to the rule of L'Hôpital, $\lim_{n\to\infty} \frac{\log_{10}(n)}{n^{0.1}} = \lim_{n\to\infty} \frac{1/(x\ln(10))}{0.1x^{-0.9}} = 0$. By the definition of the limit there exists for every $\varepsilon > 0$ an n_0, so that $\log_{10}(n) \leq \varepsilon n^{0.1}$ for all $n \geq n_0$. Hence, we can choose *any* c (e.g. $c = 1$) and from the definition of the limit follows that there is a n_0 such that $\log_{10}(n) \leq n^{0.1}$ for all $n \geq n_0$. Therefore it also holds that $\log_{10}(n) \in \mathcal{O}(n^{0.1})$.

Note: One could criticize that the above argument is not constructive. But we can observe, that $\log_{10}(10^{10}) = (10^{10})^{0.1} = 10$, and for $n > 10^{10}$ it holds that $\log_{10}(n) < n^{0.1}$. So a concrete choice for the constants is $c = 1$ and $n_0 = 10^{10}$.

Solution 1.2 *Simplifying Expressions in $\mathcal{O}$ Notation.*

a) $\mathcal{O}(2n + 14n^2) = \mathcal{O}(n^2)$

b) $\mathcal{O}(\log n + 3\sqrt{n}) = \mathcal{O}(\sqrt{n})$

c) $\mathcal{O}(16 \log_7(n^2)) = \mathcal{O}(\log(n^2)) = \mathcal{O}(2 \log n) = \mathcal{O}(\log n)$

d) $\mathcal{O}(6^{10} + \log^5(n) + \frac{n}{10}) = \mathcal{O}(n)$

Solution 1.3 *Proofs about $\mathcal{O}$ Notation.*

a) The statement is true. It follows directly from the definitions of $\mathcal{O}(g)$ and $\Omega(f)$, as it is

$$\begin{aligned} f \in \mathcal{O}(g) &\Leftrightarrow \exists c_1 \in \mathbb{R}^+, n_0 \in \mathbb{N}\ \forall n \geq n_0 : f(n) \leq c_1 g(n) \\ &\Leftrightarrow \exists c_2 \in \mathbb{R}^+, n_0 \in \mathbb{N}\ \forall n \geq n_0 : g(n) \geq c_2 f(n) \Leftrightarrow g \in \Omega(f). \end{aligned} \tag{4}$$

We can choose $c_2 = c_1^{-1}$ as constant in the second line.

b) This is a false statement. Choose, for example, $f(n) = 2n$ and $g(n) = n^2$. Then, $f \in \mathcal{O}(g)$, but $f(1) > g(1)$.

c) Such a function exists, e.g.

$$f(n) = \begin{cases} 2^n & \text{if } n \text{ is even} \\ \sqrt{n}, & \text{if } n \text{ is odd} \end{cases}$$

We note that $f(n) \notin \mathcal{O}(n)$. Therefore $\frac{2^n}{n} \xrightarrow[n\to\infty]{} \infty$ and there exists no constant that leads to an upper bound. On the other hand, it also holds that $f(n) \notin \Omega(n)$, because $\frac{\sqrt{n}}{n} \xrightarrow[n\to\infty]{} 0$ and there exists no positive constant that leads to a lower bound.

d) The statement is true. For all $a, b \in \mathbb{N}$ it holds that $\log_a(n) = \frac{\log_b(n)}{\log_b(a)}$. We set $n_0 = 1$ and $c = (\log_b(a))^{-1}$. Then it holds for all $n \geq n_0$ that

$$\log_a(n) \leq c \log_b(n) \text{ and } \log_a(n) \geq c \log_b(n). \tag{5}$$

Therefore, $\log_a(n) \in \mathcal{O}(\log_b(n))$ and $\log_a(n) \in \Omega(\log_b(n))$ and thus $\log_a(n) \in \Theta(\log_b(n))$.

e) The statement is true. By definition

$$f_1 \in \mathcal{O}(g) \Leftrightarrow \exists c_1 \in \mathbb{R}^+, n_1 : \forall n \geq n_1 : f_1(n) \leq c_1 g(n), \tag{6}$$
$$f_2 \in \mathcal{O}(g) \Leftrightarrow \exists c_2 \in \mathbb{R}^+, n_2 : \forall n \geq n_2 : f_2(n) \leq c_2 g(n). \tag{7}$$

With $c := c_1 + c_2$ and $n_0 := \max\{n_1, n_2\}$ it holds for all $n \geq n_0$ that

$$f(n) = f_1(n) + f_2(n) \leq c_1 g(n) + c_2 g(n) = c g(n). \tag{8}$$

Therefore, $f \in \mathcal{O}(g)$.

f) The statement is false. One can choose $f_1(n) = f_2(n) = n$ and $g(n) = n$. Then $f_1, f_2 \in \mathcal{O}(g)$, but $f(n) = f_1(n) \cdot f_2(n) = n^2 \notin \mathcal{O}(n)$ (there exist no constants $c > 0$ and n_0, such that $n^2 \leq cn$ applies for all $n \geq n_0$).

Solution 1.4 *Asymptotic Growth of Functions.*

We note that $\log(2^n) = n$ and $\sqrt{n} = n^{0.5}$. Moreover, $n\log^3(n) \in \mathcal{O}(\frac{n^2}{\log(n)})$, because $\log^4(n) \in \mathcal{O}(n)$. The only (!) correct order is

$$8^{11},\ \log^5(n),\ \sqrt{n},\ \log(2^n),\ n\log^3(n),\ \frac{n^2}{\log(n)},\ n^3+7n,\ 2^n,\ n^n$$