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Datenstrukturen & Algorithmen Blatt P5 HS 17

Solution for Exercise P5.1 *Sliding token.*

The idea is to compute, for each vertex u of the graph $G = (V, E)$, a value $\mathrm{win}(u)$ which is true if and only if the next player to move can win the game when the coin is placed on vertex u.

Notice that, when the coin is on u, the next player to move can win the game whenever there is a vertex v such that $(u, v) \in E$ and $win(v)$ is false. On the contrary, if all the neighbors v of u are such that $\mathrm{win}(v)$ is true, the next player to move can not always win the game (i.e., the second player always can win).

This gives us the following formula for $\mathrm{win}(u)$:

$$\mathrm{win}(u) = \bigvee_{v:(u,v)\in E} \neg\mathrm{win}(v), \tag{1}$$

where $\neg$ denotes the logic "not" function and $\vee$ denotes the logic "or" function (the "or" of 0 terms is assumed to evaluate to false).

Since G is a directed acyclic graph, it admits a topological ordering of its vertices (which can be found in $O(|V| + |E|)$ time). The values $\mathrm{win}(u)$ $\forall u \in V$ can therefore be computed in time $O(|V| + |E|)$ by considering the vertices of G in reverse topological order (i.e., from the sinks towards the sources) and using (1).

Solution for Exercise P5.2 *Submatrix Sum.*

We want to pre-compute in $O(n^2)$ time all the values $q_{i,j} = \sum_{h=1}^{i} \sum_{k=1}^{j} m_{i,j}$ for $0 \le i, j \le n$. This can be done by noticing that, for all $0 < i, j \le n$:

$$q_{i,j} = a_{i,j} + q_{i-1,j} + q_{i,j-1} - q_{i-1,j-1},$$

where $q_{0,j}$ and $q_{i,0}$ are equal to 0 by definition.

Once all the values $q_{i,j}$ have been computed, it is possible to answer a query in constant time. Indeed, we have:

$$S(a, b, c, d) = \sum_{\substack{a\le i\le b \\ c\le j\le d}} m_{i,j} = q_{b,d} - q_{a-1,d} - q_{b,c-1} + q_{a-1,c-1}.$$