/ 20 P

Programming Task P1.

Enrollment Key: `AlgoDataExam2018`

Submission: see Section 3 of the Technical Guide

Structure

In this exercise, we implement a max-heap data structure. Your task is to complete the implementation of the following methods:

- **`buildHeap()`**: builds a heap from a given array of keys in arbitrary order.
- **`insert(`**x**`)`**: adds a new key x to the heap.
- **`deleteMax()`**: deletes the maximum key from the heap.

Most of the implementation of the max-heap data structure is already provided by the template (including the code to read the input, allocate space for the heap structure, write the state of the heap, etc).

The heap holds N keys which are integer values in the range $[-2^{31}, 2^{31} - 1]$. For simplicity, we assume that the heap will not exceed more than 100'000 keys, i.e, $N \leq 100000$.

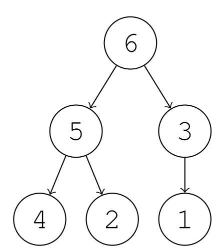
To validate the correctness of your implementation of these methods, the state of the heap is written to the output. Assuming a heap of N keys, the state of the heap is described by N partially-ordered integers that satisfy the heap property. For simplicity and convenience, the template already includes routines to output the state of the heap.

Example

- The **`buildHeap()`** method builds a heap structure by restoring the heap condition for a given array of keys. For example, given an array of $N = 5$ keys:

1	2	3	4	5	6

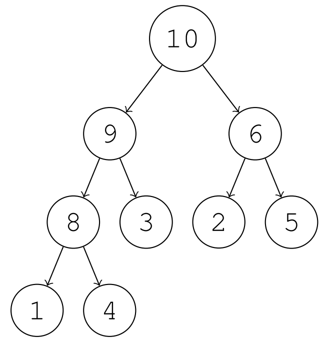
  the state of the heap, as well as the partial order on the numbers is given below:

6	5	3	4	2	1

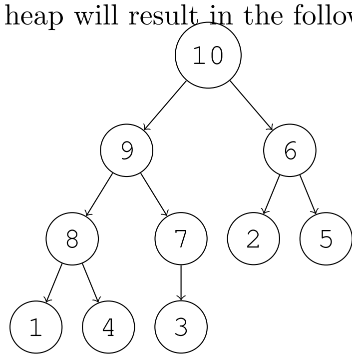
  Once the heap is built, the state of the heap is written to the output.

- For the **insert(x)** method, assume you are given the following heap of $N = 9$ keys:



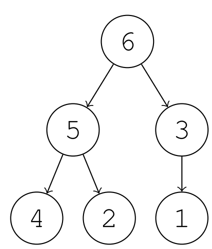
10	9	6	8	3	2	5	1	4

Inserting the key $x = 7$ into the heap will result in the following heap of $N = 10$ keys:



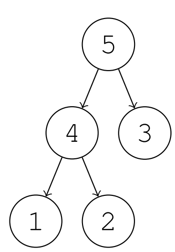
10	9	6	8	7	2	5	1	4	3

- For the **deleteMax()** method, assume you are given the following heap of $N = 6$ keys:



6	5	3	4	2	1

Once the maximum key is removed, the resulting heap of $N = 5$ keys looks as follows:



5	4	3	1	2

Grading

Overall, you can obtain a maximum of 20 judge points for this programming task. To get full points your program should require $O(n)$ time for the **`buildHeap()`** method, and $O(\log(n))$ for both **`insert(x)`** and **`deleteMax()`** methods (with reasonable hidden constants). Incomplete solutions can obtain partial points, namely:

- You can obtain up to 8 points for a correct implementation of **`buildHeap()`**.
- You can obtain up to 8 points for a correct implementation of **`insert(x)`**.
- You can obtain up to 4 points for a correct implementation of **`deleteMax()`**.

Instructions

For this exercise, we provide a program template as an Eclipse project in your workspace that helps you reading the input and writing the output. Importing any additional Java class is **not allowed** (with the exception of the already imported ones `java.io.`{`InputStream, OutputStream`} and `java.util.Scanner` class).

The project also contains data for your local testing and a `JUnit` program that runs your `Main.java` on all the local tests – just open and run `StructureTest.launch` in the project. The local test data are different and generally smaller than the data that are used in the online judge.

Submit only your `Main.java`.

The input and output are handled by the template – you should not need the rest of this text.

Input The input of this problem consists of a number of test-cases. The first line contains T, the number of test-cases. Each of the T cases is independent of the others, contains one line, and starts with a command number that can be either 1, 2 or 3.

1. If the command number is set to 1, then we read an array from the input, build a heap, and print the state of the heap on the output.

   The command number is followed by a number N, that describes the number of keys in the array, followed by N integer values in the range $[-2^{31}, 2^{31} - 1]$. All numbers are separated by a blank space.

2. If the command number is set to 2, we read in the heap, insert elements into the heap, and output the state of the heap.

   The command number is followed by a number N that describes the number of keys in the heap, followed by N integer values in the range $[-2^{31}, 2^{31} - 1]$, that satisfy the heap property. The N values are then followed by a number M that describes the number of keys that must be inserted in the heap, followed by M integers in the range $[-2^{31}, 2^{31} - 1]$. All numbers are separated by a blank space.

3. If the command number is set to 3, then we read in the heap, delete the maximum element from the heap once or several times, and output the state of the heap.

The command number is followed by a number N that describes the number of keys in the heap, followed by N integer values in the range $[-2^{31}, 2^{31}-1]$, that satisfy the heap property. The N values are then followed by a number M that describes the number of times we need to remove the maximum key from the heap. All numbers are separated by a blank space.

Output For every case, the output is the state of the heap.

The output contains one line for each test-case. More precisely, the i-th line of the output contains an array of integer values that correspond to the heap state, separated by a blank space. The output is terminated with an end-line character.

Example input:

```
3
1 6 1 2 3 4 5 6
2 9 10 9 6 8 3 2 5 1 4 1 7
3 6 6 5 3 4 2 1 1
```

Example output:

```
6 5 3 4 2 1
10 9 6 8 7 2 5 1 4 3
5 4 3 1 2
```

Solution

Since the methods are both described in the lecture's slides and in the script, we only provide sample java code as a solution.

We start with the **buildHeap()** method. We assume that values are already stored in the values[] array, but they do not hold the heap condition and have arbitrary order. We need to restore the heap condition using the method below.

```java
public void buildHeap () {
    for (int i = N / 2; i >= 0; i -= 1) {
        restoreHeapCondition(i);
    }
}

private void restoreHeapCondition (int root)  {
    int m = root, l, r;
    do {
        root = m;
        l = 2 * root + 1;
        r = 2 * root + 2;
        //
        // Check for the largest / smallest
        //
        if (l < N && cmp(values[m], values[l])) m = l;
        if (r < N && cmp(values[m], values[r])) m = r;
        //
        // Perform the swap
        //
        swap(root, m);
    } while (m != root);
}
```

The **insert(x)** method inserts a value in the heap, and places it on the right positions such that the heap condition holds. To do so, we use a method **siftUp**.

```
public void insert(int value) {
    values[N] = value;
    siftUp(N);
    N += 1;
}

private void siftUp (int idx) {
    int parent = (idx - 1) / 2;
    while (idx != 0 && cmp(values[parent], values[idx])) {
        swap(parent, idx);
        idx    = parent;
        parent = (idx - 1) / 2;
    }
}
```

Finally, the **deleteMax()** pops the first value from the heap and then restores the heap condition.

```
public void deleteMax () {
    values[0] = values[N - 1];
    N = N - 1;
    restoreHeapCondition(0);
}
```

Programming Task P2. / 20 P

Enrollment Key: `AlgoDataExam2018`

Submission: see Section 3 of the Technical Guide

Square

You are given a 2-dimensional binary matrix B having M rows and N columns ($0 \leq M, N \leq 1024$) filled with only 0's and 1's. Your task is now to find the area of a largest square submatrix that contains only 1's.

Example

$$\begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{matrix}$$

$$\downarrow$$

$$\begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & \boxed{\begin{matrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{matrix}} & & 1 \\ 1 & 1 & & & 1 \\ 1 & 0 & 0 & 1 & 0 \end{matrix}$$

In the $(M = 4) \times (N = 5)$ binary matrix illustrated above, a largest square submatrix containing only 1's has an area of 4.

Grading

Overall, you can obtain a maximum of 20 judge points for this programming task. To get full points your program should run in time $O(N \cdot M)$, with reasonable hidden constants. Slower solutions might get partial points:

- You can obtain up to 10 points for an $O(N^2 \cdot M^2)$-time solution.
- You can obtain up to 5 additional points (i.e., 15 points in total) for an $O(N \cdot M \cdot \min(M, N))$-time solution.
- You can obtain up to 5 additional points (i.e., 20 points in total) for an $O(N \cdot M)$-time solution.

Instructions

For this exercise, we provide a program template as an Eclipse project in your workspace that helps you reading the input and writing the output. Importing any additional Java class is **not allowed** (with the exception of the already imported ones `java.io.{InputStream, OutputStream}` and `java.util.Scanner` class).

The project also contains data for your local testing and a `JUnit` program that runs your `Main.java` on all the local tests – just open and run `SquareTest.launch` in the project. The local test data are different and generally smaller than the data that are used in the online judge.

Submit only your `Main.java`.

The input and output are handled by the template – you should not need the rest of this text.

Input The input of this problem consists of a number of test-cases. The first line of the input contains the number T that describes the number of test cases.

Each case is independent of the others and consists of seveal lines. The first line of each of the T test cases, contains the integers M and N separated by a space. The next M lines contain N integers either 1 or 0, separated by spaces.

Output For every case, the output should contain one integer number on a separate line – the area of a largest square submatrix that contains only 1's.

More precisely, the i-th line of the output contains a single integer corresponding to the area of the largest square found in the binary matrix of the i-th test-case.

Example input:

```
1
4 5
1 0 1 0 0
1 0 1 1 1
1 1 1 1 1
1 0 0 1 0
```

Example output:

```
4
```

Solution

We first provide a theoretical solution to the dynamic programming problem that follows the general scheme.

Dimension of the DP table:

We shall use a dynamic programming table DP that has the same size as the input matrix B, namely M rows and N columns.

Meaning of a table entry:

An entry $DP[i][j]$ corresponds to the maximum side length of the all-ones square in B that has $B[i][j]$ as its bottom right corner.

Computation of an entry (initialization and recursion):

We initialize the first row and the first column to be equal to the corresponding entry in B, since the largest square with this element as its bottom right corner can have size 1. Hence, we set $DP[i][0] = B[i][0]$ for all $0 \leq i < M$ and $DP[0][j] = B[0][j]$ for all $0 \leq j < N$.

Then, we can compute any other entry $DP[i][j]$ (with $1 \leq i < M$ and $1 \leq j < M = N$) as follows.

$$DP[i][j] = \begin{cases} \min\{DP[i-1][j], DP[i][j-1], DP[i-1][j-1]\} + 1 & \text{if } B[i][j] = 1; \\ 0 & \text{if } B[i][j] = 0 \end{cases}$$

Order of computation:

We can for instance co row by row from left to right.

Computing result:

The square of the maximum value in the DP table is the solution.

On the next page, we included a Java code snippet that illustrates an implementation of the above solution.

```
// We use a DP table of the same size as B.
// An entry DP[i][j] in the DP table corresponds to the
   maximum length of the all-ones square in B that has B[i
   ][j] as its bottom right corner.
//
int maxLength = 0
int DP[][] = new int [M][N];
//
// Initialize the first row and column
//
for (int i = 0; i < M; i++) {
    DP[i][0] = B[i][0];
}
for (int j = 0; j < N; j++) {
    DP[0][j] = B[0][j];
}
//
// Now recursively fill the DP table and remember the
   maximum entry
//
for (int i = 1; i < M; i++) {
    for (int j = 1; j < N; j++) {
        if (B[i][j] == 1) {
            DP[i][j] = Math.min(DP[i-1][j-1], Math.min(DP[i
               -1][j], DP[i][j-1]))+1;
            maxLength = Math.max(maxLength, DP[i][j]);
        }
        else {
            DP[i][j] = 0;
        }
    }
}
//
// compute the area to return
//
int maxArea = maxLength * maxLength;
```

Theory Task T1. / 14 P

Notes:

1) In this problem, you have to provide **solutions only**. You should write them directly on this sheet.
2) We assume letters to be ordered alphabetically and numbers to be ordered ascendingly, according to their values.

/ 2 P

a) Below you see four sequences of snapshots, each obtained during the execution of one of the following algorithms: `InsertionSort`, `SelectionSort`, `QuickSort`, `MergeSort`, and `BubbleSort`. For each sequence, write down the corresponding algorithm.

3	8	5	4	1	2	7	6
3	5	4	1	2	7	6	8
3	4	1	2	5	6	7	8

____`Bubble`____`Sort`

3	8	5	4	1	2	7	6
3	5	8	4	1	2	7	6
3	4	5	8	1	2	7	6

___`Insertion`___`Sort`

3	8	5	4	1	2	7	6
3	8	4	5	1	2	6	7
3	4	5	8	1	2	6	7

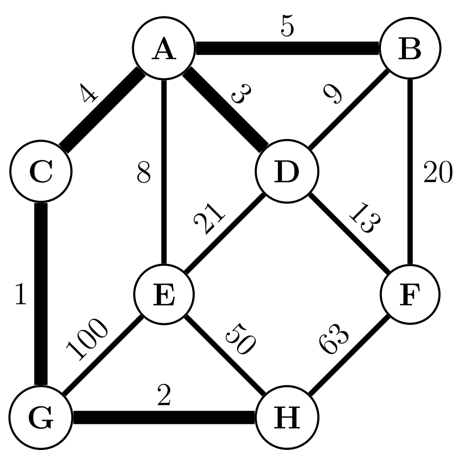
_____`Merge`_____`Sort`

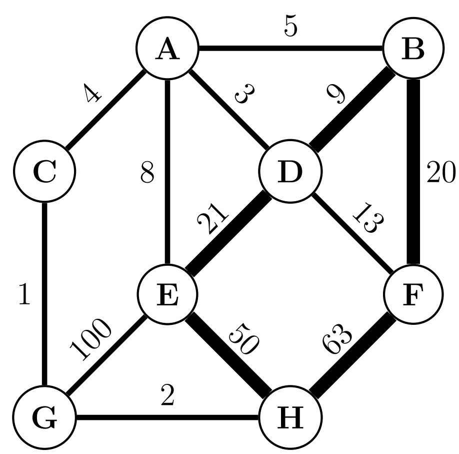
3	8	5	4	1	2	7	6
3	6	5	4	1	2	7	8
3	2	5	4	1	6	7	8

___`Selection`___`Sort`

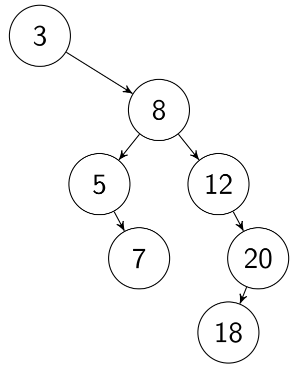
/ 2 P

b) Below you find two copies of the same graph. In the left graph, highlight the first 5 edges that Prim's algorithm chooses, when starting in vertex D. In the right graph, highlight the first 5 edges that Kruskal's algorithm discards.

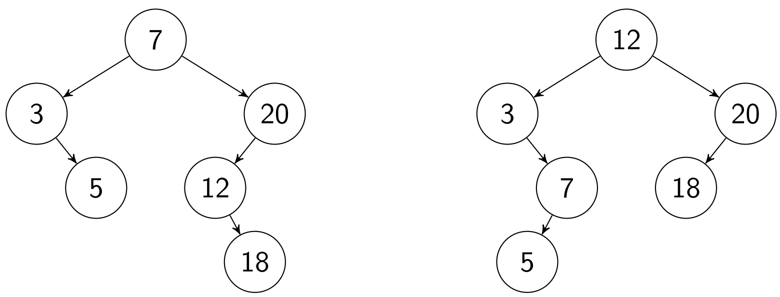




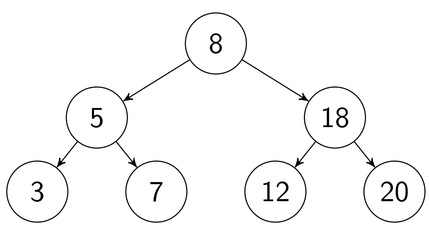
/ 1 P c) *Binary search trees*: Draw the binary search tree that is obtained when inserting in an empty tree the keys `3`, `8`, `12`, `5`, `20`, `7`, `18` in this order.



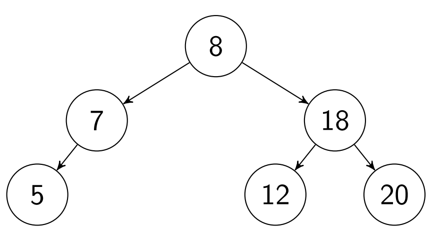
/ 1 P d) *Binary search trees*: Draw the resulting binary search tree obtained by deleting the key `8` from the following binary search tree.



/ 1 P e) *AVL-trees*: Draw the AVL-tree that is obtained when inserting in an empty tree the keys `3`, `8`, `12`, `5`, `20`, `7`, `18` in this order.

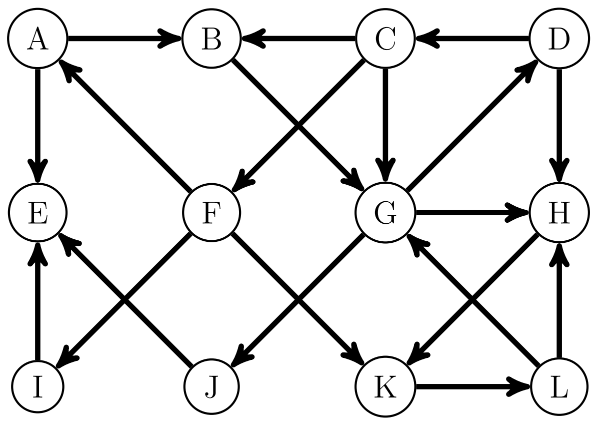


/ 1 P f) *AVL-trees*: Draw the resulting AVL-tree obtained by deleting the key `3` from the following AVL-tree.



/ 2 P

g) In the following graph, execute a depth first search and a breadth first search. Start in vertex G and assume that each traversal visits the neighbouring vertices in alphabetical order. Write down the orders (DFS-order and BFS-order) in which the vertices are visited.



DFS-order:

G, D, C, B, F, A, E, I, K, L, H, J

BFS-order:

G, D, H, J, C, K, E, B, F, L, A, I

/ 2 P

h) For each of the following claims, state whether it is true or false. You get 0.5P for a correct answer, -0.5P for a wrong answer, 0P for a missing answer. You get at least 0 points in total.

claim	**true**	**false**
$\frac{n}{\log n} \leq O(\sqrt{n})$	☐	☒
$\log n! \geq \Omega(n^2)$	☐	☒
$n^k \geq \Omega(k^n)$, if $1 < k \leq O(1)$	☐	☒
$\log_3 n^4 = \Theta(\log_7 n^8)$	☒	☐

/ 2 P i) Given the following recursion:

$$T(n) := \begin{cases} 9 \cdot T(\frac{n}{3}) + 32n - 16 & n > 1 \text{ and } n \text{ a power of } 3 \\ 6 & n = 1 \end{cases}$$

You can assume that n is a power of 3. Prove by mathematical induction that

$$T(n) = 20n^2 - 16n + 2$$

is the corresponding closed formula.

Proof by Induction:

Base Case ($n = 1$)*:* Es gilt $T(1) = 6 = 20 \cdot 1^2 - 16 \cdot 1 + 2$

Induction Hypothesis (I.H.): For any $n \geq 1$ it holds that $T(n) = 20n^2 - 16n + 2$.

Inductive step ($n \to 3n$)*:*

$$\begin{aligned} T(3n) &= 9 \cdot T(n) + 32 \cdot (3n) - 16 \\ &\overset{I.H.}{=} 9 \cdot (20n^2 - 16n + 2) + 32 \cdot (3n) - 16 \\ &= 20 \cdot (3n)^2 - 48 \cdot (3n) + 18 - 32 \cdot (3n) - 16 \\ &= 20 \cdot (3n)^2 - 16 \cdot (3n) + 2 \end{aligned}$$

Theory Task T2.

/ 14 P

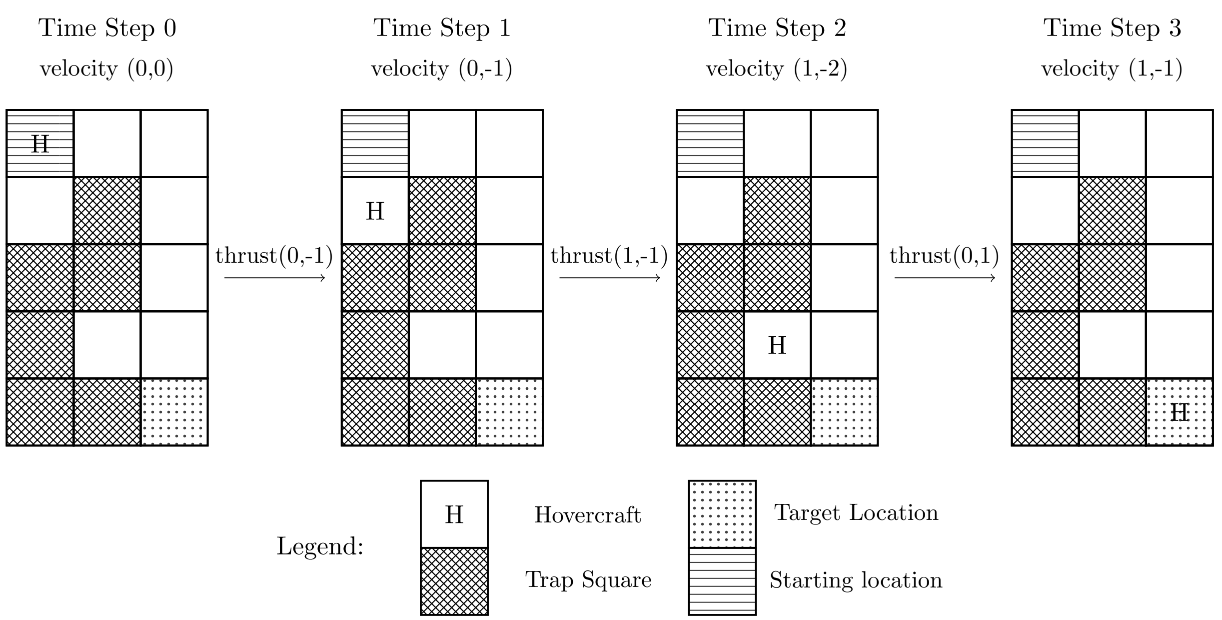
You are driving a remote-controlled hovercraft on a field. The field is a grid of $m \times n$ squares, and you want to reach a *target* location from a *starting* location. Some squares are traps and have spikes on them that can emerge from the ground and wreck your hovercraft. The locations of the trap squares are known. Also, if the hovercraft leaves the $m \times n$ field, it is destroyed. There is very little friction between your hovercraft and the ground so it is difficult to change its velocity. In order to change the hovercraft's velocity, you can use the hovercraft's thrusters.

The hovercraft has a position (x, y), $x \in \{1, \ldots, m\}$ and $y \in \{1, \ldots, n\}$, and a velocity (v_x, v_y), $v_x, v_y \in \{-2, -1, 0, 1, 2\}$. The movement of the hovercraft proceeds according to time steps. At each time step, the following actions occur (in this order):

1. The hovercraft may fire a thruster along the x axis. Then $v_x := v_x + a_x$ where $a_x \in \{-1, 1\}$. (This is only permissable if the new velocity is still valid.)
2. The hovercraft may fire a thruster along the y axis. Then $v_y := v_y + a_y$ where $a_y \in \{-1, 1\}$ (This is only permissable if the new velocity is still valid.)
3. The hovercraft's position is updated: $x := x + v_x$ and $y := y + v_y$
4. If the hovercraft is located on a trap square, the spikes emerge and the hovercraft is destroyed. The spikes then retract.

At the beginning (starting location), the hovercraft is stationary, and it can have any velocity when it reaches its destination (target location). Moreover, you can assume that it is always possible to reach the target location.

Example



/ 6 P

a) Your goal is to minimize the number of time steps it takes to get from the starting location to the target location. Model the state space as a graph $G = (V, E)$. Describe what the vertices and edges represent. For each vertex $v \in V$, precisely state to which other vertices there is an edge. Name an as efficient as possible algorithm for solving this problem, and state the running time of the algorithm in terms of m and n.

Definition of the graph (if possible, in words and not formal):

The problem can be encoded with an unweighted DAG with loops. A vertex encodes that the hovercraft ends a timestep at a specific a position on the $m \times n$ field, with the hovercraft travelling at a specific velocity. Trap squares are simply omitted from the DAG (or no outgoing edges from a trap square). Thus for each square the hovercraft can end a time step on, we encode 25 vertices, one for each possible velocity.

A vertex can be described by the tuple (x, y, v_x, v_y). A vertex (x, y, v_x, v_y) has an edge to a vertex (x', y', v'_x, v'_y) as long as

- $x' = x + v_x$
- $y' = y + v_y$
- $|v'_x - v_x| \leq 1$
- $|v'_y - v_y| \leq 1$

Number of vertices and edges (as concisely as possible in Θ notation in terms of n and m):

vertices: $\Theta(nm)$
edges: $\Theta(nm)$

Algorithm, as efficient as possible:

BFS from vertex S, where S is the vertex that corresponds to the starting location and velocity $(0, 0)$.

Running time (as concisely as possible in Θ notation in terms of n and m). Justify your answer:

Time BFS: $\Theta(|V| + |E|)$. Hence, the total running time is $\Theta(nm)$

/ 4 P

b) We now modify the problem such that the hovercraft's thrusters are powered by a cartrige of compressed carbon dioxide. The hovercraft's thrusters can only be used T times, where T is a constant. Your goal is still to minimize the number of time steps it takes to get from the starting location to the target location. For this modified problem, model the state space as a graph $G = (V, E)$, describe what the vertices and edges represent, and for each vertex $v \in V$, precisely state to which other vertices there is an edge. (Thrusting in both the x and y directions costs the same as a single thrust.) Then, name an as efficient as possible algorithm for solving this problem, and state the running time of the algorithm in terms of m and n.

Definition of the graph (if possible, in words and not formal):

Now, for each vertex of the original graph, we create $T + 1$ vertex layers. A vertex can now be described by the tuple (x, y, v_x, v_y, ℓ), where ℓ corresponds to the number of times the thrusters have been used. Thus $\ell \in \{0, \ldots, T\}$ A vertex (x, y, v_x, v_y, ℓ) has an edge to a vertex $(x', y', v'_x, v'_y, \ell')$ as long as

- $x' = x + v_x$
- $y' = y + v_y$
- $|v'_x - v_x| \leq 1$
- $|v'_y - v_y| \leq 1$
- If $v_x \neq v'_x$ or $v_y \neq v'_y$, $\ell' = \ell + 1$

Number of vertices and edges (as concisely as possible in Θ notation in terms of n and m):

\# vertices: $\Theta(nm)$
\# edges: $\Theta(nm)$

Algorithm, as efficient as possible:

BFS from vertex S, where S is the vertex that corresponds to the starting location and velocity $(0, 0)$ and level 0.

Running time (as concisely as possible in Θ notation in terms of n and m). Justify your answer:

Time BFS: $\Theta(|V| + |E|)$. Hence, the total running time is $\Theta(nm)$

/ 4 P c) We modify the original problem again. Instead of minimizing the number of time steps to reach the destination, some squares now give points when the hovercraft ends a time step on them, but most squares take points away. The goal is to score the maximum number of points, while travelling from the starting location to the end location.

Assume the following:

- You are only allowed to stop on the starting location and the target location (i.e. have velocity $(0,0)$).
- The starting location gives a negative amount of points.
- An optimum solution exists and has a finite length.

For this modified problem, model the state space as a graph $G = (V, E)$, describe what the vertices and edges represent, and for each vertex $v \in V$, precisely state to which other vertices there is an edge. Name an efficient algorithm for solving this modified problem, and state the running time of the algorithm in terms of m and n.

Definition of the graph (if possible, in words and not formal):

We again use the original graph and add weights to the edges: incoming edges to a vertex corresponds to the negative score that you get when ending a time step there.

Number of vertices and edges (as concisely as possible in Θ notation in terms of n and m):

\# vertices: $\Theta(nm)$
\# edges: $\Theta(nm)$

Algorithm, as efficient as possible:

Bellman-Ford from vertex S, where S is the vertex that corresponds to the starting location and velocity $(0,0)$.

Running time (as concisely as possible in Θ notation in terms of n and m). Justify your answer:

Time Bellman-Ford: $\Theta(|V||E|)$. Hence, the total running time is $\Theta(n^2m^2)$

Theory Task T3. / 12 P

The university curriculum consists of n courses. For simplicity assume that courses are represented by their numbers from 1 to n. Each course C has a list of prerequisite courses (this list may be empty). Before attending the course C, students have to pass at least one of its prerequisite courses. Notice that being a prerequisite is not transitive: if A is a prerequisite for B and B a prerequisite for C, then this does not imply that A is a prerequisite for C. Furthermore, you can assume that the curriculum has no circular prerequisites: there is no sequence of courses $(C_0, C_1, \ldots, C_{k-1}, C_k)$ such that for every $i < k$, C_i is a prerequisite for C_{i+1}, and also C_k is a prerequisite for C_0.

At this university, courses are very hard, so you can only pass one course each semester. Moreover, to better understand the material, you want to pass courses successively. That is, if in the previous semester you passed some course C, in the current semester you can only pass courses for which C is a prerequisite.

We call a course T *reachable* from S if there exists a sequence of courses

$$(S = C_0, C_1, \ldots, C_{k-1}, C_k = T),$$

such that for every $i < k$, C_i is a prerequisite for C_{i+1}.

In this exercise, we are interested in the following questions:

a) How can we model the courses and prerequisites as a directed graph?

b) (For a special case of the graph) is some course T reachable from some other course S?

c) (For the general case of the graph) how many different sequences are there to attend some course T after passing some course S?

In points b) and c) you can assume that the directed graph is represented by a data structure that allows you to traverse the direct successors and direct predecessors of a vertex u in time $\mathcal{O}(\deg_+(u))$ and $\mathcal{O}(\deg_-(u))$ respectively, where $\deg_-(u)$ is the in-degree of vertex u and $\deg_+(u)$ is the out-degree of vertex u.

Example

Consider the curriculum with the five courses: Linear Algebra, Multivariable Calculus, Linear Programming, Convex Optimization, Combinatorial Optimization. Linear Algebra does not have any prerequisite courses and is a prerequisite for Multivariable Calculus and Linear Programming. Multivariable Calculus and Linear Programming are prerequisites for Convex Optimization. Linear Programming is a prerequisite for Combinatorial Optimization (but Linear Algebra is not a prerequisite for Convex Optimization and Combinatorial Optimization). For an overview of prerequisites and reachability see the following table:

<table>
<tr><th>Course</th><th>Has Prerequisites</th><th>Is reachable from</th></tr>
<tr><td>Linear Algebra</td><td>—</td><td>Linear Algebra</td></tr>
<tr><td>Multivariable Calculus</td><td>Linear Algebra</td><td>Multivariable Calculus
Linear Algebra</td></tr>
<tr><td>Linear Programming</td><td>Linear Algebra</td><td>Linear Programming
Linear Algebra</td></tr>
<tr><td>Convex Optimization</td><td>Multivariable Calculus
Linear Programming</td><td>Convex Optimization
Multivariable Calculus
Linear Algebra
Linear Programming</td></tr>
<tr><td>Combinatorial Optimization</td><td>Linear Programming</td><td>Combinatorial Optimization
Linear Programming
Linear Algebra</td></tr>
</table>

Note that the number of different sequences from Linear Algebra to Convex Optimization is two: One sequence is *(Linear Algebra, Multivariable Calculus, Convex Optimization)*, the other sequence is *(Linear Algebra, Linear Programming, Convex Optimization)*.

/ 1 P

a) Model the n courses and its prerequisites as a directed graph: give a precise description of the vertices and edges of this graph $G = (V, E)$ involved (if possible, in words and not formal).

The graph $G = (V, E)$ is the following: V is a set of courses, and for two different courses C and D, $(C, D) \in E$ iff C is a prerequisite for D.

/ 4 P

b) Suppose that there exists an introductory course I which has no prerequisites and each course is reachable from I. Moreover, assume that there exists a unique possibility to pass every course after passing I if you pass courses successively, that is, for every course J there exists a unique sequence of courses $(I = C_0, C_1, \ldots, C_{k-1}, C_k = J)$, such that for every $i < k$, C_i is a prerequisite for C_{i+1}.

Using the graph as defined in a), provide an as efficient as possible algorithm that takes as input the graph G and prepocesses it so that queries "is D reachable from C?" can be answered in $\mathcal{O}(1)$ time (that is, this time does not depend on the size of the graph). State the running time of your preprocessing algorithm.

Hint: *Think about the following questions: What is the number of vertices and edges in the graph? How does the graph look like?*

Preprocessing algorithm:

In this case, G is a tree with root I. We preprocess G as follows: run DFS from vertex I and compute *pre* and *post* values for each vertex.

$\mathcal{O}(1)$**-time query algorithm:**

After preprocessing, given a query "is D reachable from C?", we answer YES, if $pre(D) \geq pre(C)$ and $post(D) \leq post(C)$, and NO otherwise.

Running time of preprocessing algorithm (as concisely as possible in Θ notation in terms of n). Justify your answer:

DFS with computing *pre* and *post* values takes time $\Theta(|E|+|V|)$. Since G is a tree, the number of vertices is n and the number of edges is $n-1$. Thus, the running time of the preprocessing algorithm is $\Theta(n)$.

(To answer the queries we need only two comparisons, so it takes $\mathcal{O}(1)$ time.)

/ 7 P

c) Suppose that you want to find the number $N(S,T)$ of possibilities to attend a course T after passing a course S if you pass courses successively. That is, $N(S,T)$ is the number of different sequences of courses $(S = C_0, C_1, \ldots, C_{k-1}, C_k = T)$, such that for every $i < k$, C_i is a prerequisite for C_{i+1}.

Provide an as efficient as possible *dynamic programming* algorithm that takes as input the graph G from task a) and the courses S and T, and outputs $N(S,T)$. Address the following aspects in your solution and state the runnung time of your algorithm:

1) *Definition of the DP table: What are the dimensions of the table $DP[\ldots]$?* What is the meaning of each entry?
2) *Computation of an entry*: How can an entry be computed from the values of other entries? Which entries do not depend on others?
3) *Calculation order*: In which order can entries be computed so that values needed for each entry have been determined in previous steps?
4) *Extracting the solution*: How can the final solution be extracted once the table has been filled?

Hint: *Think about the following question: Which graph property is implied by the sentence "The curriculum has no circular prerequisites."?*

Size of the DP table / Number of entries:

The DP-table is one dimensional, its size is n.

Meaning of a table entry:

$DP[\mathrm{C}]$ is the number of different paths from S to C in G.

Computation of an entry (initialization and recursion):

$DP[S] = 1$. And we initialize all other entries $DP[C] = 0$, for $C \neq S$.

Then, we can compute the number of paths from S to C ($C \neq S$) as follows:

$$DP[C] = \sum_{(D,C)\in E} DP[D].$$

Order of computation:

Since the curriculum has no circular prerequisites, G is acyclic, and we can topologically sort the vertices of G. We can compute $DP[C]$ in topological order of the vertices.

Computing $N(S,T)$:

$DP[T] = N(S,T)$ is a solution.

Running time in concise Θ-notation in terms of n and m, where m is the number of edges in G. Justify your answer:

Topological ordering takes time $\Theta(|E| + |V|)$. The computation of each $DP[C]$ takes time proportional to the number of incoming edges of C. The total number of incoming edges of all vertices is $|E| = m$, number of vertices is $|V| = n$, so we can compute $DP[T] = N(S,T)$ in time $\Theta(n + m)$.