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Algorithms & Data Structures — Exercise sheet 8 — HS 21

Exercise Class (Room & TA): ______________________

Submitted by: ______________________

Peer Feedback by: ______________________

Points: _____

Submission: On Monday, 22 November 2021, hand in your solution to your TA *before* the exercise class starts. Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

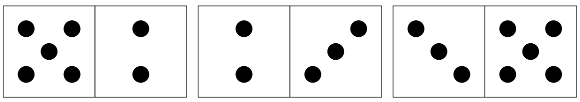
Exercise 8.1 *Party & Beer & Party & Beer* **(1 point)**.

For your birthday, you organize a party and invite some friends over at your place. Some of your friends bring their partners, and it turns out that in the end everybody (including yourself) knows exactly 7 other people at the party (note that the relation of knowing someone is commutative, i.e. if you know someone then this person also knows you and vice versa). Show that there must be an even number of people at your party.

Exercise 8.2 *Domino.*

a) A domino set consists of all possible $\binom{6}{2} + 6 = 21$ different tiles of the form $[x|y]$, where x and y are numbers from $\{1, 2, 3, 4, 5, 6\}$. The tiles are symmetric, so $[x|y]$ and $[y|x]$ is the same tile and appears only once.

   Show that it is impossible to form a line of all 21 tiles such that the adjacent numbers of any consecutive tiles coincide.



b) What happens if we replace 6 by an arbitrary $n \geq 2$? For which n is it possible to line up all $\binom{n}{2} + n$ different tiles along a line?

Exercise 8.3 *Graph connectivity.*

In this exercise, you will need to prove or find counterexamples to some statements about the connectivity of graphs. We first need to introduce/recall a few definitions.

Definition 1. A *cycle* is a sequence of vertices $v_1, \dots, v_k, v_{k+1}$ with $k \geq 3$ such that all $v_1, \dots, v_k$ are distinct, $v_1 = v_{k+1}$ and such that any two consecutive vertices are adjacent. We say that such a cycle has length k.

Definition 2. A graph is *connected* if there is a walk between every pair of vertices. It is called *disconnected* otherwise.

Definition 3. A vertex v in a connected graph is called a *cut vertex* (or *articulation point*) if the subgraph obtained by removing v (and all its incident edges) is disconnected.

Definition 4. An edge e in a connected graph is called a *cut edge* (or *bridge*) if the subgraph obtained by removing e (but keeping all the vertices) is disconnected.

In the following, we always assume that the original graph is connected. Prove or find a counterexample to the following statements:

a) If a vertex v is part of a cycle, then it is not a cut vertex.

b) If a vertex v is not a cut vertex, then v must be part of a cycle.

c) If an edge e is part of a cycle (i.e. e connects two consecutive vertices in a cycle), then it is not a cut edge.

d) If an edge e is not a cut edge, then e must be part of a cycle.

e) If u and v are two adjacent cut vertices, then the edge $e = \{u, v\}$ is a cut edge.

f) If $e = \{u, v\}$ is a cut edge, then u and v are cut vertices. What if we add the condition that u and v have degree at least 2 ?

Definition 5. We say that a graph G is *Eulerian* if it contains an Eulerian circuit (Eulerzyklus).

Definition 6. A graph $G = (V, E)$ is *bipartite* if it is possible to partition the vertices in two sets V_1 and V_2 (i.e. $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$) such that every edge $\{u, v\} \in E$ has one endpoint in V_1 and the other in V_2.

Theorem 1. *A graph is bipartite if and only if it does not contain any cycle of odd length.*

Exercise 8.4* *Equivalent characterization of bipartite graphs.*

Prove Theorem 1 above.

Exercise 8.5 *Bipartite graphs, Eulerian graphs and painting rooms* **(2 points)**.

In this exercise, you can use Theorem 1 above (even if you haven't solved exercise 8.4).

a) Prove or disprove the following statements:

   (i) Every graph G that is bipartite and Eulerian must have an even number of edges.

   (ii) Every Eulerian graph G that has an even number of vertices must also have an even number of edges.

b) You recently moved in with your best friend (see floor plan below) and you would like to repaint the room walls. Every room should be painted either in red or in purple (as these are your favorite colors), and you also would like that whenever you walk from a room to another room through a door, the color changes. Is that possible?

Note that there are 7 rooms (i.e. the Hallway, the Bathroom and the Kitchen are counted as rooms).