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Algorithms & Data Structures — Exercise sheet 12 — HS 21

Exercise Class (Room & TA): ______________________

Submitted by: ______________________

Peer Feedback by: ______________________

Points: _____

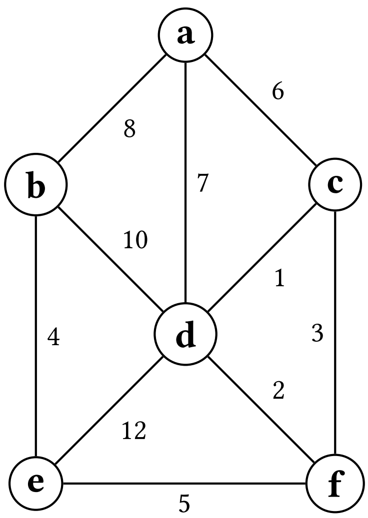
Submission: On Monday, 20. December 2021, hand in your solution to your TA *before* the exercise class starts. Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

Remark. Let $G = (V, E)$ be a weighted graph with nonnegative weights ($w(e) \geq 0 \quad \forall e \in E$) such that all edge-weights are different ($\forall e \neq e'$ in E, $w(e) \neq w(e')$). Then the minimum spanning tree of G is unique.

You can use this fact without further justification for solving this exercise sheet.

Exercise 12.1 *MST practice* ***(1 point)***.

Consider the following graph



a) Compute the minimum spanning tree (MST) using Boruvka's algorithm. For each step, provide the set of edges that are added to the MST.

b) Provide the order in which Kruskal's algorithm adds the edges to the MST.

c) Provide the order in which Prim's algorithm (starting at vertex **d**) adds the edges to the MST.

Exercise 12.2 *Minimum Spanning Tree and Shortest Paths **(1 point)**.*

Let $G = (V, E)$ be a connected edge-weighted graph where all the weights are nonnegative and distinct. Let T be a minimum spanning tree of G.

Let $v \in V$ be some vertex and define T_v to be the tree of shortest paths that is obtained by applying Dijkstra's algorithm on G starting from the source v.

Is it possible that T and T_v do not have any edge in common ? If the answer is yes, provide an example showing that it is possible. Otherwise, prove that it is impossible

Exercise 12.3 *Constructing a Fiber Optic Network **(1 point)**.*

The government of Atlantis put you in charge of installing a fiber optic network that connects all its n cities. There are two technologies of fibre optic that you can use:

- Fibre 1.0: It is a good reliable technology that is relatively cheap. There is a list of pairs of cities between which it is possible to install a direct Fibre 1.0 link. Furthermore, for each such pair, there is a corresponding positive integer cost.
- Fibre 2.0: It is an emerging technology that it extremely good and can directly connect any two cities. However, its cost is too high and the government cannot afford a single Fibre 2.0 link.

Note that all direct links are two-directional. The installed network should connect all the cities of Atlantis: Between any two cities, there should be a connected path of direct links in the network that connects them.

A philanthropist volunteered to donate the cost of exactly $k < n$ direct Fibre 2.0 links, and you can use them to connect any k pairs of cities. Your goal is to minimize the cost that is paid by the government for the Fibre 1.0 links that are needed to construct a connected network. Describe an algorithm that finds the network that costs the government the minimum amount of money.

Note that it is possible to construct a network connecting all the cities of Atlantis using only Fibre 1.0 links, but we would like to benefit from the k Fibre 2.0 links that were donated by the philantropist in order to minimize the cost that is paid by the government.

Hint: *Modify Kruskal's algorithm.*

Exercise 12.4 *Ancient Kingdom of Macedon.*

The ancient Kingdom of Macedon had n cities and m roads connecting them, such that from one city, you can reach all other $n-1$ cities. All roads were *roman roads* i.e. stone-paved roads that did not require any maintenance and no two roads were of the same length. With the technological developments in the Roman Kingdom, a new type of carriage was developed, called the *Tesla Carriage*, which was much faster than all the alternatives in the Ancient Macedon Kingdom. However, the Tesla Carriage required *asphalt roads* to operate, and such roads had to be maintained every year, or otherwise the asphalt would wear off, rendering the road unusable as if it was a roman road.

With the effort to modernize the kingdom, Phillip II promised the Ancient Macedonians that he will provide them with asphalt roads by paving some of the existing roman roads, such that every two cities can be reached through a Tesla Carriage. The price to pave a roman road or maintain an asphalt road is equal, and is proportional to the length of the road. To save money Phillip II decided to pave sufficient roman roads to fulfill his promise, while minimizing the overall yearly maintenance price.

Even in the first years, the new Tesla Carriages improved the lives of the average Ancient Macedonians, but at the same time, they also provided means for robbers to commit crimes and escape to another city. To resolve this, Phillip II decided to create checkpoints the second year, one at each asphalt road. Each of the checkpoint will have a fixed cost for both building and maintenance.

Assuming a fixed price k for each checkpoint, does Phillip II have to consider paving new roman roads, or he can maintain the same set of roads in order to make sure that the overall maintenance price of the roads and the checkpoints is still minimal? Prove your reasoning, or provide a counter example.

Note: For simplicity, assume that the roman roads were paved all at once, on the first day of the year, and maintenance will be done the same day next year, again all at once. Also assume that checkpoints can also be built at once for all roads, as well a they can be maintained all at once in a day.

Exercise 12.5* *Spanning Forest with 2 components.*

Let $G = (V, E)$ be a connected edge-weighted graph in which all weights of the edges are different and positive. Consider the following two algorithms, which take G, the weights of all edges, and two different vertices $u, v \in V$ as input.

Algorithm 1

Run Kruskal's algorithm to get a minimum spanning tree $T = (V, E_T)$.
Find the unique path π from u to v in T.
Find the edge e of maximal weight among edges in π.
Remove e from T to get a graph $H = (V, E_T \setminus \{e\})$.
return H

Algorithm 2

$M \leftarrow \{u, v\}$
$E_M \leftarrow \emptyset$
while $M \neq V$ **do**
  $\Delta M = \{\{w_0, w_1\} \in E : w_0 \in M, w_1 \in V \setminus M\}$
  Find the edge $\{w_0, w_1\} \in \Delta M$ of minimal weight.
  $E_M \leftarrow E_M \cup \{\{w_0, w_1\}\}$
  $M \leftarrow M \cup \{w_1\}$
return $H = (M, E_M)$

Prove that the two algorithms return the same graph.

Hint: *Consider the graph G' which is obtained from G by adding an edge e_0 of weight 0 between u and v (if the edge $\{u, v\}$ already exists, then simply decrease its weight to 0). Try to relate the two given algorithms on G to algorithms that you know from the lecture on G'.*