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# Algorithms & Data Structures — Exercise sheet 1 — HS 21

The solutions for this sheet are submitted at the beginning of the exercise class on October 4th.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points. In this sheet, the only exercises that are counted towards bonus points are: 1.2 and 1.4.(a–f).

You can use results from previous parts without solving those parts. For example, you can solve Exercise 1.4.b without solving 1.4.a (and use the result of 1.4.a in your solution for 1.4.b).

**Exercise 1.1** *Sum of Squares.*

Prove by mathematical induction that for every positive integer $n$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

**Solution:**

- **Base Case.**
  Let $n = 1$. Then:
  $$1 = \frac{1 \cdot (1+1) \cdot (2+1)}{6} = 1.$$

- **Induction Hypothesis.**
  Assume that the property holds for some positive integer $k$. That is,
  $$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

- **Inductive Step.**
  We must show that the property holds for $k+1$. Let's add $(k+1)^2$ to both sides of our inductive hypothesis.
  $$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}. \end{aligned}$$

  By the principle of mathematical induction, this is true for any positive integer $n$.

**Exercise 1.2** *Geometric Progression* **(1 point)**.

a) Let $a \neq 1$ be a real number. Prove by mathematical induction that for every non-negative integer $n$,

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}.$$

**Solution:**

- **Base Case.**
  Let $n = 0$. Then:

$$a^0 = 1 = \frac{a^1 - 1}{a-1}.$$

- **Induction Hypothesis.**
  Assume that the property holds for some non-negative integer $k$. That is,

$$\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}.$$

- **Inductive Step.**
  We must show that the property holds for $k+1$. Let's add $a^{k+1}$ to both sides of the induction hypothesis.

$$\begin{aligned}\sum_{i=0}^{k} a^i + a^{k+1} &= \frac{a^{k+1}-1}{a-1} + a^{k+1} \\ &= \frac{a^{k+1}-1+a^{k+1}(a-1)}{a-1} \\ &= \frac{a^{k+1}-1+a^{k+2}-a^{k+1}}{a-1} \\ &= \frac{a^{k+2}-1}{a-1}.\end{aligned}$$

  By the principle of mathematical induction, the statement is true for any non-negative integer $n$.

b) Why does your proof fail for $a = 1$?

**Solution:** The proof does not work for $a = 1$ since $a - 1 = 0$, and already the base case is not true: the left hand side is 1, while the right hand side is not defined.

**Exercise 1.3** *Asymptotic Growth.*

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that $f$ *grows asymptotically slower than* $g$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. If this is the case, we also say that $g$ *grows asymptotically faster than* $f$.

a) Prove or disprove the following statements:

1) $n^2$ grows asymptotically slower than $3n^4 + n^2 + n$.

**Solution:** We have

$$0 \le \lim_{n\to\infty} \frac{n^2}{3n^4 + n^2 + n} \le \lim_{n\to\infty} \frac{n^2}{3n^4} = \lim_{n\to\infty} \frac{1}{3n^2} = 0,$$

hence $n^2$ grows asymptotically slower than $3n^4 + n^2 + n$.

2) $\log_7(n^8)$ grows asymptotically slower than $\log_2(n^{\sqrt{n}})$.

**Solution:** We have

$$\lim_{n\to\infty} \frac{\log_7(n^8)}{\log_2(n^{\sqrt{n}})} = \lim_{n\to\infty} \frac{8\log_7(n)}{\sqrt{n}\log_2(n)} = \lim_{n\to\infty} \frac{8\log_2(n)/\log_2(7)}{\sqrt{n}\log_2(n)} = \lim_{n\to\infty} \frac{8}{\log_2(7)\sqrt{n}} = 0,$$

hence $\log_7(n^8)$ grows asymptotically slower than $\log(n^{\sqrt{n}})$.

3) $n^{1/3}$ grows asymptotically slower than $\frac{n}{\log n}$.

**Solution:** We consider the limit

$$\lim_{x\to\infty} \frac{x^{1/3}}{\frac{x}{\log x}} = \lim_{x\to\infty} \frac{\log(x)}{x^{2/3}}$$

and apply L'Hôpital's rule to obtain

$$\lim_{x\to\infty} \frac{(\log(x))'}{(x^{2/3})'} = \lim_{x\to\infty} \frac{1/x}{(2/3)x^{-1/3}} = \lim_{x\to\infty} \frac{3}{2x^{2/3}} = 0.$$

Hence, $n^{1/3}$ grows asymptotically slower than $\frac{n}{\log n}$.

4) $\sum_{k=0}^{n} k$ grows asymptotically faster than $n \log n$.

***Hint:*** *You can use the formula from Exercise 0.1.a.*

**Solution:** We have

$$\begin{aligned}\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k}{n\log n} &= \lim_{n\to\infty} \frac{n(n+1)/2}{n\log n} \\ &= \lim_{n\to\infty} \frac{n+1}{2\log n} = +\infty.\end{aligned}$$

Hence, $\sum_{k=0}^{n} k$ grows asymptotically faster than $n \log n$.

b) Place the following functions in order such that if $f$ appears before $g$, it means that $f$ grows asymptotically slower than $g$.

$$n^2 + 2n + 1, \;\; \ln(n^n) \cdot \sum_{k=0}^{n} k, \;\; \frac{n}{\ln n}, \;\; \sqrt{n}\ln(n), \;\; n\ln(n^2), \;\; \sum_{k=0}^{n} k^2, \;\; n\ln(n^n)$$

***Hint:*** *You can use formulas from Exercise 0.1.a and Exercise 1.1.*

**Solution:**

$$\sqrt{n}\ln(n), \;\; \frac{n}{\ln n}, \;\; n\ln(n^2), \;\; n^2 + 2n + 1, \;\; n\ln(n^n), \;\; \sum_{k=0}^{n} k^2, \;\; \ln(n^n) \cdot \sum_{k=0}^{n} k,$$

which can be straightforwardly obtained by considering the simplified expressions

$$\sqrt{n}\ln(n),\quad \frac{n}{\ln n},\quad 2n\ln(n),\quad n^2+2n+1,\quad n^2\ln(n),\quad \frac{n(n+1)(2n+1)}{6},\quad (n\ln n)\cdot\frac{n(n+1)}{2},$$

and applying Theorem 1 from Exercise sheet 0.

c) Let $n \geq 1$ be a positive integer. The *factorial* of $n$, denoted by $n!$, is the product of all positive integers less than or equal to $n$, that is, $n! = 1 \cdot 2 \cdots (n-1) \cdot n$.

   Prove the following statements about $n!$:

   1) $n! \leq n^n$.

      **Solution:** By definition we have $n! = n(n-1)\cdots 1 \leq n \cdot n \cdots n = n^n$.

   2) $\ln(n!)$ grows asymptotically slower than $n(\ln n)^2$.

      **Solution:** We have

$$0 \leq \lim_{n\to\infty} \frac{\ln(n!)}{n(\ln n)^2} \leq \lim_{n\to\infty} \frac{\ln(n^n)}{n(\ln n)^2} = \lim_{n\to\infty} \frac{n\ln n}{n(\ln n)^2} = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

      Therefore, $\ln(n!)$ grows asymptotically slower than $n(\ln n)^2$.

   3) $\left(\frac{n}{2}\right)^{n/2} \leq n!$.

      **Solution:** We have $n! = n(n-1)\cdots\lceil n/2\rceil\cdots 1 \geq n(n-1)\cdots\lceil n/2\rceil$, where $\lceil x\rceil$ is the smallest integer $m \geq x$. So there are $\lceil n/2\rceil$ factors that are at least $\lceil n/2\rceil \geq n/2$. Hence $n! \geq (\frac{n}{2})^{n/2}$.

   4) $\ln(n!)$ grows asymptotically faster than $n$.

      **Solution:** By the monotonicity of the logarithm we have

$$\ln(n!) \geq \ln\left(\left(\frac{n}{2}\right)^{n/2}\right) = \frac{n}{2}(\ln n - \ln 2),$$

      which implies that,

$$0 \leq \lim_{n\to\infty} \frac{n}{\ln(n!)} \leq \lim_{n\to\infty} \frac{n}{\frac{n}{2}(\ln n - \ln 2)} = \lim_{n\to\infty} \frac{2}{\ln n - \ln 2} = 0.$$

      Therefore, $\ln(n!)$ grows faster than $n$.

**Exercise 1.4** *Pasture Break* **(2 points)**.

A cow is imprisoned in a farm and wishes to escape. The fence of the farm contains a hole but the cow does not know where it is exactly located. At time $t = 0$, the cow is at position $x_{\text{cow}} = 0$, and the hole is at a positive integer distance $k$ from the cow, but it can be either on the left or on the right of the cow, i.e., the hole is located either at $x_{\text{hole}} = +k$ or $x_{\text{hole}} = -k$. At time $t = 0$, the cow starts searching for the hole from its initial position at $x_{\text{cow}} = 0$, and wishes to find the hole as quickly as possible.

The cow can move at a constant speed of 1: For example, if it moves to the right, then at time $t = 2$ the cow will be at position $x = +2$. Since the cow does not know whether the hole is on the left or on the right, it decided to alternately explore increasingly larger areas to the left and to the right. More precisely, let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of all positive integers. The cow chooses a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$, and then follows the following procedure:

- First, it moves to the left until it reaches $x = -f(1)$.
- Then, it switches direction and starts moving to the right until it reaches $x = f(2)$.
- Then, it switches direction and starts moving to the left until it reaches $x = -f(3)$.
- Then, it switches direction and starts moving to the right until it reaches $x = f(4)$.
- And so on …
- The cow follows this search procedure until it finds the hole.

Let $T(x_{\text{hole}})$ be the time taken to finish the search assuming that the hole is at position $x_{\text{hole}}$. We define:

$$T_{\text{best}}(k) = \min\big\{T(-k), T(+k)\big\} \quad \text{and} \quad T_{\text{worst}}(k) = \max\big\{T(-k), T(+k)\big\}.$$

In class we analyzed $T_{\text{worst}}(k)$ for two choices of $f$: $f(n) = n$ and $f(n) = 2^{n-1}$. In this exercise, we will see what happens for other choices of $f$.

We extend the domain of definition of $f$ to non-negative integers by adopting the convention that $f(0) = 0$. We define $n_k \geq 0$ as the unique non-negative integer satisfying $f(n_k) < k \leq f(n_k + 1)$.

a) Show that

$$T_{\text{best}}(k) = k + 2\sum_{i=1}^{n_k} f(i) \quad \text{and} \quad T_{\text{worst}}(k) = k + 2\sum_{i=1}^{n_k+1} f(i).$$

It suffices to provide an informal argument.

**Solution:** Since the moving speed is equal to 1, the search time $T(k)$ is equal to the total traveled distance. Now since $f(n_k) < k \leq f(n_k + 1)$, the cow will find the hole during the $(n_k + 1)$-th or the $(n_k + 2)$-th iteration, depending on whether $x_{\text{hole}} = (-1)^{n_k-1}k$ or $x_{\text{hole}} = (-1)^{n_k}k$.

Let $m \in \{n_k+1, n_k+2\}$ be the iteration during which the cow finds the hole. The cow completes the first $m - 1$ iterations and then travels a partial distance of the $m^{th}$ iteration:

- The total traveled distance during the first $m - 1$ iterations is $f(m-1) + 2\sum_{i=1}^{m-2} f(i)$.
- The distance that the cow travels during the $m^{th}$ iteration until it finds the hole is equal to $f(m-1) + k$.

Therefore, the total traveled distance is

$$k + 2\sum_{i=1}^{m-1} f(i).$$

By replacing $m$ with $n_k+1$ (respectively, $n_k+2$), we obtain the formula of $T_{\text{best}}(k)$ (respectively, $T_{\text{worst}}(k)$).

Let us see what happens if $f$ "grows quadratically", i.e., $f(n) = n^2$.

b) Show that for every positive integer $k \geq 1$, we have

$$\frac{k\sqrt{k}}{100} \leq T_{\text{best}}(k) \leq T_{\text{worst}}(k) \leq 100k\sqrt{k}.$$

***Hint:*** *Note that*[1] $n_k = \left\lceil \sqrt{k} \right\rceil - 1$ *and use the formula from Exercise 1.1. Also, you can use the fact that the function* $g(x) := \frac{x-1}{x}$ *is strictly increasing for all* $x > 0$ *without justification.*

**Solution:** We have $n_k = \left\lceil \sqrt{k} \right\rceil - 1$. Now from a) and 1.1, we have:

$$T_{\text{best}}(k) = k + 2\sum_{i=1}^{n_k} i^2 = k + \frac{2n_k(n_k+1)(2n_k+1)}{6} = k + \frac{n_k(n_k+1)(2n_k+1)}{3}.$$

For $k = 1$, we have $n_k = 0$ and $T_{\text{best}}(k) = 1$, hence $1\sqrt{1} \leq T_{\text{best}}(1)$.

For $k > 1$, we have $n_k = \left\lceil \sqrt{k} \right\rceil - 1 \geq \sqrt{k} - 1$. Now notice that $\frac{\sqrt{k}-1}{\sqrt{k}}$ achieves its minimum when $k = 2$, and hence

$$n_k \geq \left( \frac{\sqrt{2}-1}{\sqrt{2}} \right) \sqrt{k} \geq \frac{\sqrt{k}}{4}.$$

Therefore,

$$\begin{aligned} T_{\text{best}}(k) &= k + \frac{n_k(n_k+1)(2n_k+1)}{3} \\ &\geq \frac{\frac{\sqrt{k}}{4}\left(\frac{\sqrt{k}}{4}\right)\left(\frac{2\sqrt{k}}{4}\right)}{3} \\ &= \frac{k\sqrt{k}}{96}. \end{aligned}$$

We conclude that for every $k \geq 1$, we have $\frac{1}{96}k\sqrt{k} \leq T_{\text{best}}(k)$.

Similarly, from a) and 1.1, we have:

$$T_{\text{worst}}(k) = k + 2\sum_{i=1}^{n_k+1} i^2 = k + \frac{2(n_k+1)(n_k+2)(2n_k+3)}{6} = k + \frac{(n_k+1)(n_k+2)(2n_k+3)}{3}.$$

We have $n_k = \left\lceil \sqrt{k} \right\rceil - 1 \leq \sqrt{k}$, which implies that

$$\begin{aligned} T_{\text{worst}}(k) &= k + \frac{(n_k+1)(n_k+2)(2n_k+3)}{3} \\ &\leq k\sqrt{k} + \frac{\left(\sqrt{k}+\sqrt{k}\right)\left(\sqrt{k}+2\sqrt{k}\right)\left(2\sqrt{k}+3\sqrt{k}\right)}{3} \\ &= 11k\sqrt{k}. \end{aligned}$$

We conclude that for every $k \geq 1$, we have

$$\frac{1}{96}k\sqrt{k} \leq T_{\text{best}}(k) \leq T_{\text{worst}}(k) \leq 11k\sqrt{k}.$$

**Remark.** Intuitively, this means that if $f(n) = n^2$, then the asymptotic growth of $T_{\text{best}}(k)$ and $T_{\text{worst}}(k)$ is similar to that of $k\sqrt{k}$. More generally, if $f(n) = n^\ell$ for some $\ell > 0$, then it is possible to show that the asymptotic growth of $T_{\text{best}}(k)$ and $T_{\text{worst}}(k)$ is similar to that of $k^{1+\frac{1}{\ell}}$. This means that for every function $f(n)$ that "grows polynomially in $n$", both $T_{\text{best}}(k)$ and $T_{\text{worst}}(k)$ grow asymptotically faster than $k$.

[1] Here $\lceil x \rceil$ denotes the ceiling of $x$, i.e., it is the smallest integer $\ell$ satisfying $\ell \geq x$.

In parts c)–f) we consider functions of "exponential growth", i.e., $f(n) = e_a(n) := a^{n-1}$ for some positive integer $a \geq 2$.

c) What is the worst-case search time $T_{\text{worst}}(k)$? (In terms of $k$ and $n_k$).

*Hint: Use the formula from Exercise 1.2.*

**Solution:** from a) and 1.2, we have:

$$T_{\text{worst}}(k) = k + 2\sum_{i=1}^{n_k+1} a^{i-1} = k + 2\sum_{i=0}^{n_k} a^i = k + \frac{2(a^{n_k+1}-1)}{a-1}.$$

Notice that if the cow knew the correct direction (i.e., the sign of $x_{\text{hole}}$), then the optimal search time would be $k$. Therefore, we would like to compare $T_{\text{worst}}(k)$ to $k$. The quotient $\frac{T_{\text{worst}}(k)}{k}$ represents the worst-case relative delay that is incurred by the search procedure compared to the optimal search time.

For non-negative integer $N$, let $R_{\text{worst}}(N)$ be the maximum value of the worst-case relative delay $\frac{T_{\text{worst}}(k)}{k}$ over all $k$ that satisfy $n_k = N$.

d) What is the value of $R_{\text{worst}}(N)$?

***Hint:*** *For fixed $n_k = N$, the value of $\frac{T_{\text{worst}}(k)}{k}$ achieves its maximum when $k = a^{N-1} + 1$.*

**Solution:** From c) we have $T_{\text{worst}}(k) = k + \frac{2(a^{n_k+1}-1)}{a-1}$. Therefore,

$$\frac{T_{\text{worst}}(k)}{k} = 1 + \frac{2(a^{n_k+1}-1)}{(a-1)k}.$$

Since for fixed $n_k = N$, the value of $\frac{T_{\text{worst}}(k)}{k}$ achieves its maximum when $k = a^{N-1}+1$, the maximum value of $\frac{T_{\text{worst}}(k)}{k}$ is equal to

$$1 + \frac{2(a^{N+1}-1)}{(a-1)(a^{N-1}+1)}.$$

The *asymptotic worst-case relative delay* is defined as $\lim_{N\to\infty} R_{\text{worst}}(N)$.

e) Compute the asymptotic worst-case relative delay.

**Solution:**

$$\lim_{N\to\infty} R_{\text{worst}}(N) = \lim_{N\to\infty}\left(1 + \frac{2(a^{N+1}-1)}{(a-1)(a^{N-1}+1)}\right) = 1 + \frac{2a^2}{a-1}.$$

f) What is the parameter $a \geq 2$ that minimizes the asymptotic worst-case relative delay?

***Hint:*** *You can use the fact that the function $g(x) := \frac{x^2}{x-1}$ is strictly increasing for $x \geq 2$ without justification.*

**Solution:** For positive integers $a \geq 2$, the function $a \mapsto \frac{2a^2}{a-1} + 1$ is strictly increasing, and minimized when $a = 2$.

Let us now consider a function $f$ that grows at a doubly exponential rate, e.g., $f(n) = d(n) := 2^{2^{n-1}}$.

g)* Show that $T_{\text{best}}(k) \leq 5k$ for $k \geq 10$.

**Solution:** We have $n_k = \lceil \log_2(\log_2(k)) \rceil \leq \log_2(\log_2(k)) + 1$. Now from a), we have

$$\begin{aligned} T_{\text{best}}(k) &= k + 2\sum_{i=1}^{n_k} 2^{2^{i-1}} \leq k + 2(n_k - 1)2^{2^{n_k-2}} + 2 \cdot 2^{2^{n_k-1}} \\ &\leq k + 2(\log_2(\log_2(k)))2^{2^{\log_2(\log_2(k))-1}} + 2 \cdot 2^{2^{\log_2(\log_2(k))}} \\ &= k + 2(\log_2(\log_2(k)))2^{\frac{1}{2}\log_2(k)} + 2k = 3k + 2(\log_2(\log_2(k)))\sqrt{k}. \\ &\leq 5k, \end{aligned}$$

where the last inequality follows from the fact that $(\log_2(\log_2(k)))\sqrt{k} \leq k$ for all $k \geq 10$.

h)* Show that there exists $C > 0$ and a strictly increasing sequence $(k_\ell)_{\ell \geq 1}$ satisfying $k_\ell \in \mathbb{N}$ and $T_{\text{worst}}(k_\ell) \geq C k_\ell^2$ for every $\ell \geq 1$.

**Solution:** We have $n_k = \lceil \log_2(\log_2(k)) \rceil$. Therefore, if $k_\ell = 2^{2^{\ell-1}} + 1$, then $n_{k_\ell} = \ell$.

Now from a), we have

$$T_{\text{worst}}(k_\ell) = k_\ell + 2\sum_{i=1}^{\ell+1} 2^{2^{i-1}} \geq k_\ell + 2^{2^\ell} = k_\ell + (k_\ell - 1)^2 = k_\ell^2 - k_\ell + 1 \geq k_\ell^2 - k_\ell.$$

Now since $k_\ell \geq k_1 = 3$, we have $k_\ell^2 - k_\ell = k_\ell^2 \left(1 - \frac{1}{k_\ell}\right) \geq k_\ell^2 \left(1 - \frac{1}{3}\right) = \frac{2k_\ell^2}{3}$. Therefore,

$$T_{\text{worst}}(k_\ell) \geq \frac{2k_\ell^2}{3}.$$