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04 October 2021

Algorithms & Data Structures — Exercise sheet 2 — HS 21

The solutions for this sheet are submitted at the beginning of the exercise class on October 11th.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points. In this sheet, the only exercises that are counted towards bonus points are: 2.2, 2.4 and 2.5 (a, e, f). You can use results from previous parts without solving those parts.

Exercise 2.1 *Induction.*

1. Prove via mathematical induction that for all integers $n \geq 5$,

$$2^n > n^2 \,.$$

Solution:

- **Base Case.**
  Let $n = 5$. Then:
$$2^5 = 32 > 25 = 5^2 \,.$$
- **Induction Hypothesis.**
  Assume that the property holds for some positive integer k. That is,
$$2^k > k^2 \,.$$
- **Inductive Step.**
  We must show that the property holds for $k + 1$.
$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &\overset{\text{I.H.}}{>} 2 \cdot k^2 \\ &= k^2 + k^2 \\ &\geq k^2 + 5k \\ &= k^2 + 2k + 3k \\ &\geq k^2 + 2k + 15 \\ &> k^2 + 2k + 1 \\ &= (k+1)^2 \,. \end{aligned}$$

  By the principle of mathematical induction, this is true for every positive integer n.

2. Let x be a real number. Prove via mathematical induction that for every positive integer n, we have
$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \,,$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}\,.$$

We use a standard convention $0! = 1$, so $\binom{n}{0} = \binom{n}{n} = 1$ for every positive integer n.

Hint: *You* can use the following fact without justification: for every $1 \leq i \leq n$,

$$\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}\,.$$

Solution: We will use the identity from the hint to show (via mathematical induction) that

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i\,.$$

- **Base Case.**
  Let $n = 1$. Then $(1+x)^1 = \binom{1}{0}x^0 + \binom{1}{1}x^1 = \sum_{i=0}^{n}\binom{n}{i}x^i$.
- **Induction Hypothesis.**
  Assume that the property holds for some positive integer k. That is,

$$(1+x)^k = \sum_{i=0}^{k} \binom{k}{i} x^i.$$

- **Inductive Step.**
  We must show that the property holds for $k+1$.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)(1+x)^k \\
&\overset{I.H.}{=} (1+x)\sum_{i=0}^{k}\binom{k}{i}x^i \\
&= \Big(\sum_{i=0}^{k}\binom{k}{i}x^i\Big) + \Big(\sum_{i=0}^{k}\binom{k}{i}x^{i+1}\Big) \\
&= \Big(\sum_{i=0}^{k}\binom{k}{i}x^i\Big) + \Big(\sum_{i=1}^{k+1}\binom{k}{i-1}x^i\Big) \\
&= \binom{k}{0}x^0 + \sum_{i=1}^{k}\Big(\binom{k}{i}x^i + \binom{k}{i-1}x^i\Big) + \binom{k}{k}x^{k+1} \\
&= \binom{k+1}{0}x^0 + \sum_{i=1}^{k}\binom{k+1}{i}x^i + \binom{k+1}{k+1}x^{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}x^i.
\end{aligned}$$

  By the principle of mathematical induction, this is true for every positive integer n.

Asymptotic Notation

When we estimate the number of elementary operations executed by algorithms, it is often useful to ignore constant factors and instead use the following kind of asymptotic notation, also called O-Notation. We denote by $\mathbb{R}^+$ the set of all (strictly) positive real numbers and by $\mathbb{N}$ the set of all (strictly) positive integers.

Definition 1 (O-Notation)**.** Let $n_0 \in \mathbb{N}$, $N := \{n_0, n_0+1, \ldots\}$ and let $f : N \to \mathbb{R}^+$. $O(f)$ is the set of all functions $g : N \to \mathbb{R}^+$ such that there exists $C > 0$ such that for all $n \in N$, $g(n) \leq Cf(n)$.

In general, we say that $g \leq O(f)$ if Definition 1 applies after restricting the domain to *some* $N = \{n_0, n_0+1, \ldots\}$. Some sources use the notation $g = O(f)$ or $g \in O(f)$ instead.

Instead of working with this definition directly, it is often easier to use limits in the way provided by the following theorem.

Theorem 1 (Theorem 1.1 from the script)**.** *Let $f : N \to \mathbb{R}^+$ and $g : N \to \mathbb{R}^+$.*

- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f \leq O(g)$ and $g \not\leq O(f)$.*
- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C \in \mathbb{R}^+$, then $f \leq O(g)$ and $g \leq O(f)$.*
- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $f \not\leq O(g)$ and $g \leq O(f)$.*

The theorem holds all the same if the functions are defined on $\mathbb{R}^+$ instead of N. In general, $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is the same as $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ if the second limit exists.

The following theorem can also be helpful when working with O-notation.

Theorem 2. *Let $f, g, h : \mathbb{N} \to \mathbb{R}^+$. If $f \leq O(h)$ and $g \leq O(h)$, then*

1. *For every constant $c \geq 0$, $c \cdot f \leq O(h)$.*
2. *$f + g \leq O(h)$.*

Notice that for all real numbers $a, b > 1$, $\log_a n = \log_a b \cdot \log_b n$ (where $\log_a b$ is a positive constant). Hence $\log_a n \leq O(\log_b n)$. So you don't have to write bases of logarithms in asymptotic notation, that is, you can just write $O(\log n)$.

Exercise 2.2 *Comparison of functions* **(1 point)**.

Prove or disprove the following statements:

a) $(n^2 - n + 1)^2 \leq O(n^4)$ and $n^4 \leq O((n^2 - n + 1)^2)$.

Solution:

$$\lim_{n\to\infty} \frac{(n^2 - n + 1)^2}{n^4} = \lim_{n\to\infty} \frac{n^4 - 2n^3 + 3n^2 - 2n + 1}{n^4} = \lim_{n\to\infty} \left(1 - \frac{2}{n} + \frac{3}{n^2} - \frac{2}{n^3} + \frac{1}{n^4}\right) = 1 \in \mathbb{R}^+,$$

so by Theorem 1 we have $(n^2 - n + 1)^2 \leq O(n^4)$ and $n^4 \leq O((n^2 - n + 1)^2)$.

b) $\sqrt{n} \leq O(\sqrt[3]{n \log n})$.

Solution:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt[3]{n \log n}} = \lim_{n\to\infty} \frac{n^{1/2}}{n^{1/3} \log^{1/3} n} = \lim_{n\to\infty} \frac{n^{1/6}}{\log^{1/3} n} = \infty,$$

so by Theorem 1 we have $\sqrt{n} \not\leq O(\sqrt[3]{n \log n})$.

c) $\log_{100}^2(n) \leq O(\log_2(n^{100}))$.

Solution: Note that $\log_{100}(n) = \frac{\ln n}{\ln 100}$ and $\log_2(n^{100}) = 100 \log_2 n = \frac{100 \ln n}{\ln 2}$. Therefore,

$$\lim_{n\to\infty} \frac{\log_{100}^2(n)}{\log_2(n^{100})} = \lim_{n\to\infty} \frac{\ln^2 n / \ln^2 100}{100 \ln n / \ln 2} = \lim_{n\to\infty} \frac{\ln 2 \ln^2 n}{100 \ln^2 100 \ln n} = \frac{\ln 2}{100 \ln^2 100} \lim_{n\to\infty} \ln n = \infty,$$

so by Theorem 1 we have $\log_{100}^2(n) \not\leq O(\log_2(n^{100}))$.

d) $\sum_{k=1}^n \left(k^2 e^k + \ln^3 k\right) \leq O(3^n)$.

Solution: Since $k \mapsto k^2 e^k$ and $k \mapsto \ln^3 k$ are increasing functions, we have

$$\sum_{k=1}^{n} \left(k^2 e^k + \ln^3 k\right) \leq n \left(n^2 e^n + \ln^3 n\right) = n^3 e^n + n \ln^3 n.$$

Therefore,

$$0 \leq \lim_{n\to\infty} \frac{\sum_{k=1}^n \left(k^2 e^k + \ln^3 k\right)}{3^n} = \lim_{n\to\infty} \frac{n^3 e^n + n \ln^3 n}{3^n} = \lim_{n\to\infty} \left(\frac{n^3}{(3/e)^n} + \frac{n \ln^3 n}{3^n}\right).$$

Clearly $\lim_{n\to\infty} \frac{n \ln^3 n}{3^n} = 0$. Moreover, since $3/e > 1$, we also have $\lim_{n\to\infty} \frac{n^3}{(3/e)^n} = 0$. Therefore, $\lim_{n\to\infty} \frac{\sum_{k=1}^n \left(k^2 e^k + \ln^3 k\right)}{3^n} = 0$ and thus $\sum_{k=1}^n \left(k^2 e^k + \ln^3 k\right) \leq O(3^n)$ by Theorem 1.

e) $\sum_{k=0}^n 2^k \leq O(2^n)$.

Solution: By Exercise 1.2, we have

$$\sum_{k=0}^{n} 2^k = \frac{2^{n+1} - 1}{2 - 1} = 2^{n+1} - 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n 2^k}{2^n} = \lim_{n\to\infty} \frac{2^{n+1} - 1}{2^n} = \lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) = 2 \in \mathbb{R}^+,$$

so by Theorem 1 we have $\sum_{k=0}^n 2^k \leq O(2^n)$.

Exercise 2.3 *Asymptotic growth of* $\ln(n!)$.

Recall that the factorial of a positive integer n is defined as $n! = 1 \times 2 \times \cdots \times (n-1) \times n$.

a) Show that $\ln(n!) \leq O(n \ln n)$.

Hint: *You can use the result of Exercise 1.3.c.1*

Solution: From 1.3.c.1, we have $n! \leq n^n$, which implies that $\ln(n!) \leq n \ln n$ and thus $\ln(n!) \leq O(n \ln n)$.

b) Show that $n \ln n \leq O(\ln(n!))$.

Hint: *You can use the result of Exercise 1.3.c.3*

Solution: From 1.3.c.3, we have $n! \geq \left(\frac{n}{2}\right)^{n/2}$. Now by the monotonicity of the logarithm we have

$$\ln(n!) \geq \ln\left(\left(\frac{n}{2}\right)^{n/2}\right) = \frac{n}{2}\left(\ln n - \ln 2\right),$$

so $n \ln n \leq 2 \ln(n!) + 2 \ln 2$. By Theorem 1, $n \ln n \leq O(\ln(n!))$.

Exercise 2.4 *Runtime of Iterative Algorithms* **(1 point)**.

Many algorithms are iterative in the sense that they repeat n iterations of some procedure. The number of iterations n is usually monotonically related to the size of the input, i.e., bigger inputs require more iterations. Furthermore, it is often the case that later iterations take more time. More precisely, if $t(k)$ is the time taken by the k-th iteration, then $t(i) \leq t(j)$ for all $i \leq j$. Clearly, the total running time $T(n)$ of such an algorithm satisfies

$$T(n) = \sum_{k=1}^{n} t(k).$$

In this exercise, we are interested in analyzing the asymptotic growth of $T(n)$ in terms of that of $t(n)$. As we previously mentioned, in this exercise we always assume that the function $t : \mathbb{N} \to \mathbb{N}$ is nondecreasing.

a) Show that for arbitrary nondecreasing function $t : \mathbb{N} \to \mathbb{N}$, we always have $T(n) \leq O(n \cdot t(n))$.

Solution: We have

$$T(n) = \sum_{k=1}^{n} t(k) \leq \sum_{k=1}^{n} t(n) = n \cdot t(n),$$

where the inequality follows from the fact that t is nondecreasing. Therefore, $T(n) \leq O(n \cdot t(n))$.

b) Assume that $t(n)$ grows polynomially, i.e., there exist real numbers $\beta > 0$ and $C \geq 1$ such that for all $n \in \mathbb{N}$, $\frac{1}{C} \cdot n^{\beta} \leq t(n) \leq C \cdot n^{\beta}$. Show that $n \cdot t(n) \leq O(T(n))$.

Hint: *Use the fact that for every $n \geq 2$, we have $\sum_{k=\lceil \frac{n}{2} \rceil}^{n} k^{\beta} \geq \frac{n}{2} \cdot \left(\frac{n}{2}\right)^{\beta}$, where $\lceil x \rceil$ denotes the smallest integer ℓ satisfying $\ell \geq x$.*

Solution:

For all $n \geq 2$, we have:

$$n \cdot t(n) \overset{(*)}{\leq} n \cdot C \cdot n^{\beta} = 2^{\beta+1} C \cdot \left(\frac{n}{2}\right) \cdot \left(\frac{n}{2}\right)^{\beta} \leq 2^{\beta+1} C \cdot \sum_{k=\lceil \frac{n}{2} \rceil}^{n} k^{\beta}$$

$$\overset{(\dagger)}{\leq} 2^{\beta+1} C \cdot \sum_{k=\lceil \frac{n}{2} \rceil}^{n} C \cdot t(k) \leq 2^{\beta+1} C^2 \cdot \sum_{k=1}^{n} t(k) = 2^{\beta+1} C^2 \cdot T(n),$$

where in $(*)$ we used the fact that $t(n) \leq C \cdot n^{\beta}$, and in $(\dagger)$ we used the fact that $\frac{1}{C} \cdot k^{\beta} \leq t(k)$. We conclude that $n \cdot t(n) \leq O(T(n))$.

c) Show that for arbitrary nondecreasing function $t : \mathbb{N} \to \mathbb{N}$, we always have $t(n) \leq O(T(n))$.

Solution: We have

$$t(n) \leq \sum_{k=1}^{n} t(k) = T(n),$$

which means that $t(n) \leq O(T(n))$.

d) Assume that $t(n)$ grows exponentially, i.e., there exist real numbers $\alpha > 1$ and $C \geq 1$ such that for all $n \in \mathbb{N}$, $\frac{1}{C} \cdot \alpha^n \leq t(n) \leq C \cdot \alpha^n$. Show that $T(n) \leq O(t(n))$.

Hint: *Use Exercise 1.2.*

Solution: For all $n \geq 1$, we have

$$T(n) = \sum_{k=1}^{n} t(k) \overset{(*)}{\leq} \sum_{k=1}^{n} C \cdot \alpha^k \leq C \cdot \sum_{k=0}^{n} \alpha^k = C \cdot \frac{\alpha^{n+1} - 1}{\alpha - 1}$$
$$\leq \frac{C}{\alpha - 1} \cdot \alpha^{n+1} = \frac{C\alpha}{\alpha - 1} \cdot \alpha^n \overset{(\dagger)}{\leq} \frac{C\alpha}{\alpha - 1} \cdot C \cdot t(n) = \frac{C^2\alpha}{\alpha - 1} \cdot t(n),$$

where in $(*)$ we used the fact that $t(k) \leq C \cdot \alpha^k$, and in $(\dagger)$ we used the fact that $\frac{1}{C} \cdot \alpha^n \leq t(n)$. We conclude that $n \cdot t(n) \leq O(T(n))$. We conclude that $T(n) \leq O(t(n))$.

Remark. Together, the results of a) and b) imply that if $t(n)$ grows polynomially, then $T(n)$ has the same asymptotic growth as $n \cdot t(n)$. It is possible to show a similar result if $t(n)$ is bounded or grows logarithmically. The results of c) and d) imply that if $t(n)$ grows exponentially, then $T(n)$ has the same asymptotic growth as $t(n)$.

Exercise 2.5 *The Euclidean Algorithm For Finding the Greatest Common Divisor* **(1 point)**.

We say that $a \in \mathbb{N}$ divides $b \in \mathbb{N}$ if there exists $q \in \mathbb{N}$ such that $b = aq$. The greatest common divisor $\gcd(n, m)$ of $n \in \mathbb{N}$ and $m \in \mathbb{N}$ is the greatest integer $d \in \mathbb{N}$ that divides both n and m. In this exercise, we are interested in computing $\gcd(n, m)$ as efficiently as possible.

Note that since $\gcd(n, m) = \gcd(m, n)$, we may assume without loss of generality that $n \geq m$. So in this exercise we always assume $n \geq m$.

The simplest algorithm that we can think of is the following procedure:

- Check whether $d = m$ is a common divisor for n and m. If yes, output d. Otherwise, go to the next step.
- Check whether $d = m - 1$ is a common divisor for n and m. If yes, output d. Otherwise, go to the next step.
- Then, we similarly check $m - 2, m - 3, \ldots, 1$.

Let $T_{\text{simple}}(n, m)$ be the number of iterations (steps) that are taken by the above simple algorithm to find the greatest common divisor.

a) For fixed $n \geq 1$, determine $\min\limits_{1 \leq m \leq n} T_{\text{simple}}(n, m)$ and $\max\limits_{1 \leq m \leq n} T_{\text{simple}}(n, m)$?

Hint: *You may use the following fact without proof: For every positive integer n, $\gcd(n, n-1) = 1$.*

Solution: For general $n \geq m > 0$, we have

$$T_{\text{simple}}(n, m) = m - \gcd(n, m) + 1.$$

We observe the following facts:

- For $m = n$, it is easy to see that $\gcd(n, n) = n$, hence $T_{\text{simple}}(n, n) = 1$.
- For $m = n - 1$, we have $\gcd(n, n - 1) = 1$, and hence $T_{\text{simple}}(n, n - 1) = n - 1$.

- For every $1 \leq m < n-1$, we have $1 \leq \gcd(n,m) \leq m$ and hence $1 \leq T_{\text{simple}}(n,m) \leq m < n-1$.

We conclude that

$$\min_{1 \leq m \leq n} T_{\text{simple}}(n,m) = 1 \quad \text{and} \quad \max_{1 \leq m \leq n} T_{\text{simple}}(n,m) = n-1.$$

In the remaining of this exercise, we will describe an algorithm that can compute $\gcd(n,m)$ much more efficiently. This algorithm is based on applying divisions with remainders. For every $a, b \in \mathbb{N}$, there exist unique nonnegative integers q, r, such that $b = aq + r$ and $0 \leq r < a$. The number q (respectively, r) is called *the quotient (respectively, the remainder) of the division of b by a.*

Let q and r be the quotient and remainder of the division of n by m, i.e., $n = qm + r$ and $0 \leq r < m$.

b)* Show that if $r = 0$, then $\gcd(n,m) = m$.

Solution: If $r = 0$, then $n = qm$, and so m is a common divisor of both n and m. Since there cannot be a divisor of m that is greater than m, we conclude that $\gcd(n,m) = m$.

c)* Show that if $r > 0$, then $\gcd(n,m) = \gcd(m,r)$.

Solution: The fact that $n = mq + r$ implies that every common divisor of m and r is also a divisor of n. On the other hand, the fact that $r = n - mq$ implies that every common divisor of n and m is also a divisor of r. Therefore, we must have $\gcd(n,m) = \gcd(m,r)$.

The above two properties suggest the following algorithm to compute $\gcd(n,m)$:

- Define $n_0 = n$ and $n_1 = m$.
- For every $i \geq 1$, as long as, $n_i > 0$, we perform the division with remainder of n_{i-1} by n_i, i.e., we find the unique q_i, r_i that satisfy $n_{i-1} = q_i n_i + r_i$ and $0 \leq r_i < n_i$, and then we define $n_{i+1} = r_i$.
- When we reach i for which $n_i = 0$, we stop and output n_{i-1}.

This is called the Euclidean algorithm for computing the greatest common divisor.

d)* Show that the above algorithm correctly computes the greatest common divisor of n and m.

Solution: We will show by induction that for every $i \geq 1$ for which n_i is defined, we have $\gcd(n_{i-1}, n_i) = \gcd(n,m)$.

- **Base Case.**
  For $i = 1$, we have $\gcd(n_0, n_1) = \gcd(n,m)$ because $n_0 = n$ and $n_1 = m$.
- **Induction Hypothesis.**
  Assume that the property holds for some positive integer i. That is,

$$\gcd(n_{i-1}, n_i) = \gcd(n,m).$$

- **Inductive Step.**
  We must show that the property holds for $i+1$. If n_{i+1} is defined, it must be the remainder of the Euclidean division of n_{i-1} by n_i. It follows from c) that $\gcd(n_{i-1}, n_i) = \gcd(n_i, n_{i+1})$. On the other hand, from the induction hypothesis we have $\gcd(n_{i-1}, n_i) = \gcd(n,m)$. Therefore, $\gcd(n_i, n_{i+1}) = \gcd(n,m)$.

By the principle of mathematical induction, we deduce that for every $i \geq 1$ for which n_i is defined, we have $\gcd(n_{i-1}, n_i) = \gcd(n, m)$. Now if $n_i = 0$, it follows from b) that $\gcd(n_{i-1}, n_i) = \gcd(n_{i-1}, 0) = n_{i-1}$, hence $\gcd(n, m) = n_{i-1}$.

This proves that if the algorithm outputs something, then the output must be correct. It remains to show that the algorithm must stop after a finite number of steps and cannot loop forever. This can be seen from the fact that the remainder of the Euclidean division is smaller than the divisor. Hence, for every $i \geq 1$, we have $n_{i+1} < n_i$, which means that we will reach 0 after at most m steps.

e) Show that for every $i \geq 2$, we have $n_i < \frac{n_{i-2}}{2}$.

Solution: Recall that n_i is the remainder of the Euclidean division of n_{i-2} by n_{i-1}. We distinguish two cases:

- If $n_{i-1} \leq \frac{n_{i-2}}{2}$, then we use the fact that the remainder of the Euclidean division is always smaller than the divisor. Hence, $n_i < n_{i-1} \leq \frac{n_{i-2}}{2}$.
- If $n_{i-1} > \frac{n_{i-2}}{2}$, then

$$n_{i-2} - n_{i-1} < n_{i-2} - \frac{n_{i-2}}{2} = \frac{n_{i-2}}{2} < n_{i-1}.$$

By noticing that we also have $n_{i-2} = n_{i-1} + (n_{i-2} - n_{i-1})$ and using the fact that the pair (q_{i-1}, r_{i-1}) of the Euclidean division is unique, we deduce that $q_{i-1} = 1$ and $r_{i-1} = n_{i-2} - n_{i-1}$. Now since $n_i = r_{i-1}$ by definition, we conclude that

$$n_i = r_{i-1} = n_{i-2} - n_{i-1} < \frac{n_{i-2}}{2}.$$

Let $T_E(n, m)$ be the number of iterations that are taken by the Euclidean algorithm to find the greatest common divisor of n and m (where $n \geq m$).

f) Show that $T_E(n, m) \leq O(\log(n))$.

Solution: From e) we can easily show by induction on $k \geq 1$ that as long as n_{2k} is defined, then we must have $n_{2k} < \frac{n_0}{2^k} = \frac{n}{2^k}$. In particular, if $k = \lceil \log_2(n) \rceil \geq 1$, then $n_{2k} < \frac{n}{n} = 1$. This means that if n_{2k} is defined, then it must be 0. Therefore,

$$T_E(n, m) \leq 2\lceil \log_2(n) \rceil \leq 2(\log_2(n) + 1) \leq 4 \log_2(n).$$

g)* Compute the greatest common divisor of 139472305678615 and 273426584985 using the Euclidean algorithm. If you had used the "naive simple algorithm" that we mentioned at the beginning of the exercise, would you have been able to compute the GCD of these two numbers using only pen and paper?

Solution: The sequence $(n_i)_i$ is as follows:

- $n_0 = 139472305678615$.
- $n_1 = 273426584985$.
- $n_2 = 24747336265$.
- $n_3 = 1205886070$.
- $n_4 = 629614865$.

- $n_5 = 576271205$.
- $n_6 = 53343660$.
- $n_7 = 42834605$.
- $n_8 = 10509055$.
- $n_9 = 798385$.
- $n_{10} = 130050$.
- $n_{11} = 18085$.
- $n_{12} = 3455$.
- $n_{13} = 810$.
- $n_{14} = 215$.
- $n_{15} = 165$.
- $n_{16} = 50$.
- $n_{17} = 15$.
- $n_{18} = 5$.
- $n_{19} = 0$.

Therefore, $\gcd(139472305678615, 273426584985) = 5$. The Euclidean algorithm needed only 18 Euclidean divisions.

If we used the naive simple algorithm, we would need $T_{\text{simple}}(139472305678615, 273426584985) = 273426584985 - 5 = 273426584980$ iterations. Even if we assume that every iteration on pen and paper takes only 1 second on average (which is not realistic), we would need more than 8000 years to finish the calculations.