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Algorithms & Data Structures — Exercise sheet 8 — HS 21

Exercise Class (Room & TA): ____________

Submitted by: ____________

Peer Feedback by: ____________

Points: ____

Submission: On Monday, 22 November 2021, hand in your solution to your TA *before* the exercise class starts. Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

Exercise 8.1 *Party & Beer & Party & Beer* **(1 point)**.

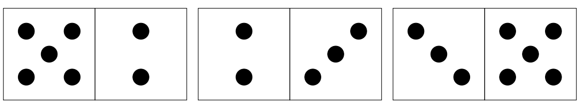
For your birthday, you organize a party and invite some friends over at your place. Some of your friends bring their partners, and it turns out that in the end everybody (including yourself) knows exactly 7 other people at the party (note that the relation of knowing someone is commutative, i.e. if you know someone then this person also knows you and vice versa). Show that there must be an even number of people at your party.

Solution: Let n denote the number of people at your party. We can model the situation by a graph $G = (V, E)$, where the vertices V are the people who came to your party, and two vertices are connected by an edge whenever they know each other. Since everybody knows exactly 7 other people at the party, we have $\deg(v) = 7$ for all vertices $v \in V$. Therefore, $\sum_{v \in V} \deg(v) = 7n$ since there are n vertices. On the other hand, by the Handshaking lemma (Handschlaglemma) we also know that $\sum_{v \in V} \deg(v) = 2|E|$, and in particular the sum of the degrees must be an even number. In other words, $7n$ is an even number, which implies that n must be even as well.

Exercise 8.2 *Domino.*

a) A domino set consists of all possible $\binom{6}{2} + 6 = 21$ different tiles of the form $[x|y]$, where x and y are numbers from $\{1, 2, 3, 4, 5, 6\}$. The tiles are symmetric, so $[x|y]$ and $[y|x]$ is the same tile and appears only once.

   Show that it is impossible to form a line of all 21 tiles such that the adjacent numbers of any consecutive tiles coincide.



b) What happens if we replace 6 by an arbitrary $n \geq 2$? For which n is it possible to line up all $\binom{n}{2} + n$ different tiles along a line?

Solution: We directly solve the general problem.

First we note that we may neglect tiles of the form $[x|x]$. If we have a line without them, then we can easily insert them to any place with an x. Conversely, if we have a line with them then we can just remove them. Thus the problem with and without these tiles are equivalent.

Consider the following graph G with n vertices, labelled with $\{1, \ldots, n\}$. We represent the domino tile $[x|y]$ by an edge between vertices x and y. Then the resulting graph G is a complete graph K_n, i.e., the graph where every pair of vertices is connected by an edge. A line of domino tiles corresponds to a walk in this graph that uses every edge at most once, and vice versa. A complete line (of *all* tiles) corresponds to an Eulerian walk in G. Thus we need to decide whether $G = K_n$ has an Euler walk or not.

K_n is obviously connected. If n is odd then all vertices have even degree $n-1$, and thus the graph is Eulerian. On the other hand, if n is even then all vertices have odd degree $n-1$. If $n \geq 4$ is even, then there are more than 3 vertices of odd degree, and therefore K_n does not have an Euler walk. Finally, for $n = 2$, the graph K_n is just an edge and has an Euler walk. Summarizing, there exists an Euler walk if $n = 2$ or n is odd, and there is no Euler walk in all other cases. Hence, it is possible to line up the domino tiles if $n = 2$ or n is odd, and it is impossible otherwise.

Exercise 8.3 *Graph connectivity.*

In this exercise, you will need to prove or find counterexamples to some statements about the connectivity of graphs. We first need to introduce/recall a few definitions.

Definition 1. A *cycle* is a sequence of vertices $v_1, \ldots, v_k, v_{k+1}$ with $k \geq 3$ such that all $v_1, \ldots, v_k$ are distinct, $v_1 = v_{k+1}$ and such that any two consecutive vertices are adjacent. We say that such a cycle has length k.

Definition 2. A graph is *connected* if there is a walk between every pair of vertices. It is called *disconnected* otherwise.

Definition 3. A vertex v in a connected graph is called a *cut vertex* (or *articulation point*) if the subgraph obtained by removing v (and all its incident edges) is disconnected.

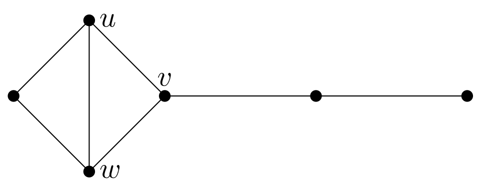
Definition 4. An edge e in a connected graph is called a *cut edge* (or *bridge*) if the subgraph obtained by removing e (but keeping all the vertices) is disconnected.

In the following, we always assume that the original graph is connected. Prove or find a counterexample to the following statements:

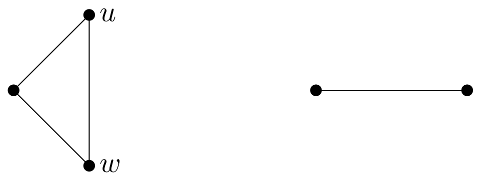
a) If a vertex v is part of a cycle, then it is not a cut vertex.

Solution:

The following graph is a counterexample:



Indeed, v is clearly part of a cycle (the triangle uvw for example), but removing v yields the following graph:

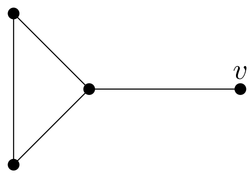


The above graph is disconnected. Hence, v is also a cut vertex.

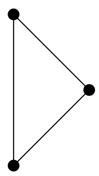
b) If a vertex v is not a cut vertex, then v must be part of a cycle.

Solution:

The following graph is a counterexample:



Indeed, v is not part of a cycle (remember that the vertices forming a cycle must be disjoint). However, removing v yields the following connected graph:



Hence, v is also not a cut vertex.
Remark. The following statement is true: If a vertex of degree at least 2 is not a cut vertex, then it must lie on a cycle. (Proof: Consider two neighbours u_1, u_2 of that vertex v. Since v is not a cut vertex, after removing v there is still a path from u_1 to u_2. Together with the edges $\{u_2, v\}$ and $\{v, u_1\}$, this forms a cycle that contains v.)

c) If an edge e is part of a cycle (i.e. e connects two consecutive vertices in a cycle), then it is not a cut edge.

Solution:

This statement is correct, and we can prove it as follows. Let G be a connected graph and let $e = \{v_1, v_2\}$ be an edge of G that is part of a cycle $v_1 \ldots v_k$ for some $k \geq 3$. To show that e is not a cut edge, we will show that any two vertices u and w of G can be joined by a walk that does not use the edge e. So consider any two vertices u and w. Since G is connected, there is a walk $uu_1 \ldots u_n w$ from u to w in G. If the walk doesn't use the edge e, then we are already done. If the walk does use edge e, this means that the vertices v_1 and v_2 must appear consecutively (at least once) in $uu_1 \ldots u_n w$. We replace every appearance of $v_1 v_2$ in the walk by the path $v_1 v_k v_{k-1} \ldots v_2$, and every apperance of $v_2 v_1$ by the same path in the other direction $v_2 v_3 \ldots v_k v_1$. This yields a walk from u to w that does not use the edge e and concludes the proof.

d) If an edge e is not a cut edge, then e must be part of a cycle.

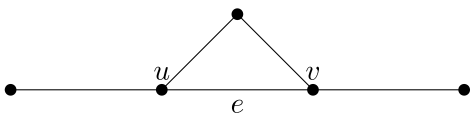
Solution:

This statement is correct, and we can prove it as follows. Let G be a connected graph and let $e = \{v_1, v_2\}$ be an edge of G that is not a cut edge. Then any pair of vertices in G can be joined by a walk that doesn't use the edge e. In particular, there is a walk from v_1 to v_2 that doesn't use e. Moreover, we can turn this walk into a path. Indeed, for any vertex u that appears more than once in the walk, we just remove the whole walk between its first appearance and its last appearance in the walk. Doing this sequentially along the walk, we obtain a path $v_1, u_1, \ldots, u_k, v_2$ from v_1 to v_2 that does not use e. In particular, this path must contain at least 3 vertices (the only way to have a path from v_1 to v_2 using 2 vertices is by using the edge e). We can then close this path with the edge e to form a cycle $v_1, u_1, \ldots, u_k, v_2, v_1$, and hence e is part of a cycle.

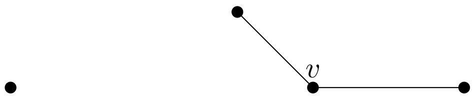
e) If u and v are two adjacent cut vertices, then the edge $e = \{u, v\}$ is a cut edge.

Solution:

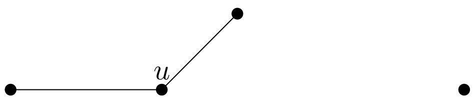
The following graph is a counterexample:



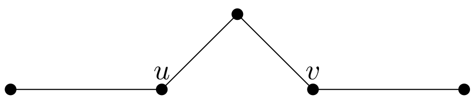
Indeed, removing u yields



while removing v yields



and both of these graphs are disconnected, which means that u and v are cut vertices. However, removing the edge $e = \{u, v\}$ yields the following connected graph:

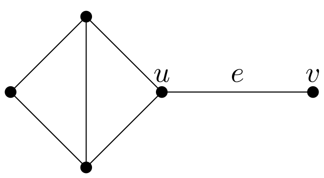


Hence, e is not a cut edge.

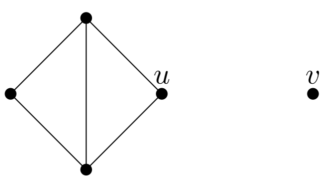
f) If $e = \{u, v\}$ is a cut edge, then u and v are cut vertices. What if we add the condition that u and v have degree at least 2 ?

Solution:

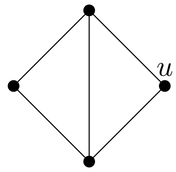
The following graph is a counterexample:



Indeed, removing e yields the following graph:



The above graph is disconnected, so e is a cut edge. However, removing v yields the following connected graph:



Hence, v is not a cut vertex.

If we add the condition that u and v have degree at least 2, then the statement is actually correct. We will only show that u is a cut vertex, since the proof that v is also a cut vertex is exactly the same with the two vertices exchanged.

Since $\deg(u) \geq 2$, u must have a neighbor $u' \neq v$ in the original graph G. We claim that after removing u and all its incident edges, there is no walk from u' to v in the obtained subgraph G', which means that it is disconnected and thus u is indeed a cut vertex.

Suppose by contradiction that there is a walk from u' to v in G'. Using the same trick as in part d), we can turn this walk into a path π from u' to v. Since $u' \neq v$, this path must use at least one edge. Since every edge incident to u in G was removed to create G', π does not use any edge incident to u, and in particular it does not use the edge e nor the edge uu'. But then we obtain a cycle in G by concatenating the path π with the edge e and then uu' (note that this cycle indeed contains at least 3 edges, and hence 3 vertices). So e is part of a cycle in G, and by part c) it therefore cannot be a cut edge. This is the desired contradiction.

Definition 5. We say that a graph G is *Eulerian* if it contains an Eulerian circuit (Eulerzyklus).
Definition 6. A graph $G = (V, E)$ is *bipartite* if it is possible to partition the vertices in two sets V_1 and V_2 (i.e. $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$) such that every edge $\{u, v\} \in E$ has one endpoint in V_1 and the other in V_2.
Theorem 1. *A graph is bipartite if and only if it does not contain any cycle of odd length.*

Exercise 8.4* *Equivalent characterization of bipartite graphs.*

Prove Theorem 1 above.

Solution:

First suppose that $G = (V, E)$ is a bipartite graph and let $V_1, V_2 \subseteq V$ be subsets of vertices that satisfy the conditions of Definition 6. Let $v_1, \ldots, v_k, v_{k+1}$ be a cycle of G of length k. Since $V = V_1 \cup V_2$, v_1 must be contained either in V_1 or in V_2. Without loss of generality, we can assume that $v_1 \in V_1$. Then since G has (V_1, V_2) as a bipartition and $\{v_1, v_2\}$ is an edge, we must have $v_2 \in V_2$. By the same reasoning, we actually have $v_{2i+1} \in V_1$ and $v_{2i} \in V_2$ for all i. But since $v_{k+1} = v_1 \in V_1$, this means that $k + 1$ must be of the form $2i + 1$ for some i, which means that $k = 2i$ is even.

Conversely, suppose that $G = (V, E)$ is a graph which does not contain any cycle of odd length. Note first that if G is not connected, it suffices to find a bipartition $(V_1(H), V_2(H))$ of each connected component H of G, since this yields a bipartition of G given by

$$V_1 = \bigcup_H V_1(H) \quad \text{and} \quad V_2 = \bigcup_H V_2(H),$$

where the unions are taken over all connected components H of G. Therefore, we can assume that G is connected.

For two vertices $u, v \in V$, we define their distance $d_G(u, v)$ as the length of (i.e the number of edges in) one of the shortest path connecting u and v. For example, $d_G(u, u) = 0$, and $d_G(u, v) = 1$ if and only if u and v are adjacent. Note that since G is connected, $d_G(u, v)$ is finite for all $u, v \in V$. Let v be an arbitrary vertex in G. We will show that the sets

$$V_1 := \{u \in V : d_G(u, v) \text{ is even}\},$$
$$V_2 := \{u \in V : d_G(u, v) \text{ is odd}\},$$

form a bipartition of G. Indeed, $V_1 \cup V_2 = V$ since $d_G(u, v)$ is finite for all $u \in V$. Moreover, $V_1 \cap V_2 = \emptyset$ because a distance $d_G(u, v)$ cannot be even and odd at the same time. It remains to show the hardest part, namely that there are no edges connecting two vertices in V_1 or two vertices in V_2. The proof is basically the same for both cases, so we just show that there cannot be an edge $\{x, y\} \in E$ with $x, y \in V_1$.

Suppose by contradiction that there exists $x, y \in V_1$ connected by an edge $e \in E$. Let P be one of the shortest path from x to v, and Q one of the shortest path from y to v. Let us denote by $\ell(S)$ the length of a path S, i.e. the number of edges in S, so that $\ell(P) = d_G(x, v)$ and $\ell(Q) = d_G(y, v)$ by definition. The paths P and Q have at least one vertex in common, namely v. So let v_1 be the first common vertex of P and Q when taking the paths from their different endpoints x and y to v. Clearly, since P and Q are shortest paths, their subpaths bewteen v_1 and v must be shortest paths between v_1 and v. In particular they are of the same length $d_G(v_1, v)$. Let P_1 be the subpath of P from x to v_1, and Q_1 the subpath of Q from y to v_1. Since $x, y \in V_1$, we have that $\ell(P) = d_G(x, v)$ and $\ell(Q) = d_G(y, v)$ are both even. Therefore, $\ell(P_1) = \ell(P) - d_G(v_1, v)$ and $\ell(Q_1) = \ell(Q) - d_G(v_1, v)$ have the same parity. Consider the cycle C created by taking P_1 from x to v_1, then Q_1 "in reverse direction" from v_1 to y and then the edge e from y to x. The length of C is given by $\ell(P_1) + \ell(Q_1) + 1$, which is odd since $\ell(P_1)$ and $\ell(Q_1)$ have the same parity. This is a contradiction to the assumption that G does not contain any cycle of odd length. Hence no two vertices of V_1 are adjacent, which concludes the proof.

Exercise 8.5 *Bipartite graphs, Eulerian graphs and painting rooms* **(2 points)**.

In this exercise, you can use Theorem 1 above (even if you haven't solved exercise 8.4).

a) Prove or disprove the following statements:

   (i) Every graph G that is bipartite and Eulerian must have an even number of edges.

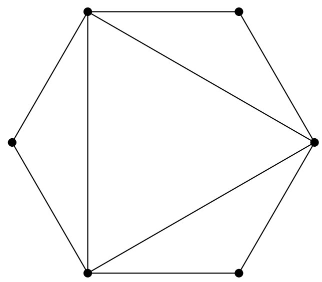
   Solution: This statement is correct, and we can prove it as follows. Let $G = (V, E)$ be a bipartite Eulerian graph and let V_1, V_2 be a bipartition of G, i.e. a partition of the vertices of G that satisfies Definition 6. Moreover, let us number the edges of the graph as $E = \{e_1, \ldots, e_m\}$ in such a way that $e_1 e_2 \ldots e_{m-1} e_m$ is an Eulerian circuit of G. Our goal is to prove that m is even.

   Since $e_1 e_2 \ldots e_{m-1} e_m$ is an Eulerian circuit, and in particular a circuit, we know that the starting vertex and ending vertex coincide, so let $v_0 \in V$ be this vertex. Since V_1 and V_2 form a partition of V, v_0 must be conatined in one of these two sets. Without loss of generality, let us assume that $v_0 \in V_1$. If we follow the Eulerian cycle, everytimne we go through an edge we will change from the set V_1 to the set V_2 or vice-versa, because every edge has one endpoint in V_1 and the other in V_2 by definition. In other words, after going through the edge e_1 we end up at a vertex $v_1 \in V_2$, and then using vertex e_2 we arrive at a vertex $v_2 \in V_1$, and so on. In particular, after following an odd number of edges from the Eulerian cycle, we stand at some vertex $u \in V_2$, and this vertex cannot be v_0 since $v_0 \in V_1$. But the last edge e_m brings us to $v_0 \in V_1$, so m cannot be an odd number, which proves that m is even.

(ii) Every Eulerian graph G that has an even number of vertices must also have an even number of edges.

Solution:

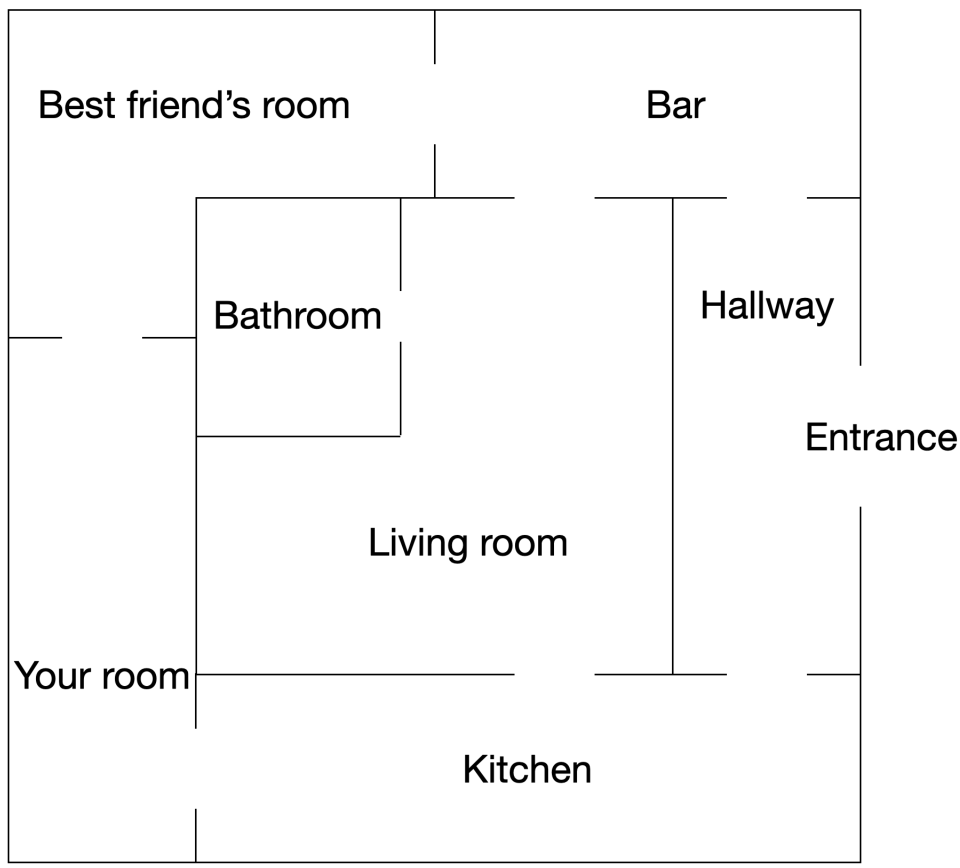
The following graph is a counterexample:



Indeed, the graph is clearly connected and one can easily check that the degree of each vertex is even, and hence the graph is Eulerian. However, it has an even number of vertices (6) but an odd number of edges (9).

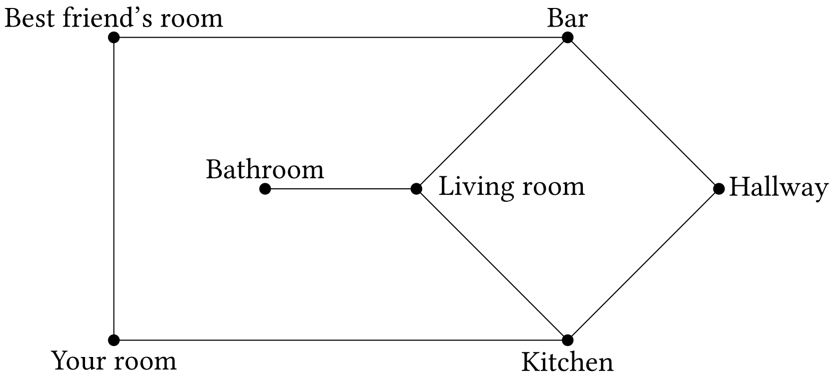
b) You recently moved in with your best friend (see floor plan below) and you would like to repaint the room walls. Every room should be painted either in red or in purple (as these are your favorite colors), and you also would like that whenever you walk from a room to another room through a door, the color changes. Is that possible?

Note that there are 7 rooms (i.e. the Hallway, the Bathroom and the Kitchen are counted as rooms).



Solution:

We first model the above floor plan as the following graph, where the vertices represent the different rooms of the flat, and two vertices are conncted with an edge whenever there is a door between the two corresponding rooms.



Note that it is possible to paint the room in the prescribed way if and only if the above graph is bipartite. Indeed, if such a painting exists, then assigning the rooms painted in red to V_1 and the other rooms (painted in purple) to V_2 gives a bipartition of the graph, because every edge will have an endpoint in V_1 and the other in V_2 as otherwise there would be two ajacent room painted in the same color. Conversely, given a bipartition of the above graph, painting all rooms in V_1 in red and all rooms in V_2 in purple will satsisfy the condition.

By Theorem 1, the above graph is not bipartite, because it contains a cycle of odd length (for example the cycle *Your room, Kitchen, Hallway, Bar, Best friend’s room, Your room*, of length 5). Therefore, it is not possible to paint the rooms in the prescribed way.