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# Algorithms & Data Structures — Exercise sheet 1 — HS 22

The solutions for this sheet are submitted at the beginning of the exercise class on 3 October 2022.

Exercises that are marked by * are "challenge exercises". They do not count towards bonus points.

You can use results from previous parts without solving those parts.

**Exercise 1.1** *Guess the formula* **(1 point)**.

Consider the recursive formula defined by $a_1 = 1$ and $a_{n+1} = 2a_n + 1$. Find a simple closed formula for $a_n$ and prove that $a_n$ follows it using induction.

***Hint:*** *Write out the first few terms. How fast does the sequence grow?*

**Solution:**

Writing out the first few terms, we get: $1, 3, 7, 15, 31$, etc. From this sequence, we guess the closed formula

$$a_n = 2^n - 1.$$

Now we prove $a_n = 2^n - 1$ by induction.

- **Base Case.**
  For $n = 1$:
  $$1 = a_1 = 2^1 - 1 = 1,$$
  so it is true for $n = 1$.
- **Induction Hypothesis.**
  Now we assume that it is true for $n = k$, i.e., $a_k = 2^k - 1$.
- **Induction Step.**
  We will prove that it is also true for $n = k + 1$.
  $$2^{k+1} - 1 = 2^{k+1} - 2 + 1 = 2^k \cdot 2 - 2 + 1 = (2^k \cdot 2 - 2) + 1 = 2(2^k - 1) + 1$$
  $$= 2a_k + 1 = a_{k+1}$$
  Hence it is true for $n = k + 1$.

**Exercise 1.2** *Sum of Squares.*

Prove by mathematical induction that for every positive integer $n$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

**Solution:**

- **Base Case.**
  Let $n = 1$. Then:
  $$1 = \frac{1 \cdot (1+1) \cdot (2+1)}{6} = 1\,.$$
- **Induction Hypothesis.**
  Assume that the property holds for some positive integer $k$. That is,
  $$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}\,.$$
- **Inductive Step.**
  We must show that the property holds for $k+1$. Let's add $(k+1)^2$ to both sides of our inductive hypothesis.
  $$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\,. \end{aligned}$$

  By the principle of mathematical induction, this is true for any positive integer $n$.

**Exercise 1.3** *Sums of powers of integers* **(1 point)**.

In this exercise, we fix an integer $k \in \mathbb{N}_0$.

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \le n^{k+1}$.

**Solution:**

As all terms in the sum are at most $n^k$, we have:

$$\sum_{i=1}^{n} i^k \le \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}.$$

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$.

***Hint:*** *Consider the second half of the sum, i.e., $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k$. How many terms are there in this sum? How small can they be?*

**Solution:**

We have:

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k = \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \cdot \left(\frac{n}{2}\right)^k$$

By definition of $\lceil \cdot \rceil$, we have $\left\lceil \frac{n}{2} \right\rceil - 1 \leq \frac{n}{2}$, hence $n - \left\lceil \frac{n}{2} \right\rceil + 1 \geq \frac{n}{2}$. Hence

$$\sum_{i=1}^{n} i^k \geq \frac{n}{2} \cdot \left(\frac{n}{2}\right)^k = \frac{1}{2^{k+1}} \cdot n^{k+1}.$$

Together, these two inequalities show that $C_1 \cdot n^{k+1} \leq \sum_{i=1}^{n} i^k \leq C_2 \cdot n^{k+1}$, where $C_1 = \frac{1}{2^{k+1}}$ and $C_2 = 1$ are two constants independent of $n$. Hence, when $n$ is large, $\sum_{i=1}^{n} i^k$ behaves "almost like $n^{k+1}$" up to a constant factor.

**Exercise 1.4** *Asymptotic growth* **(1 point)**.

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that $f$ *grows asymptotically slower than* $g$ if $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that $g$ *grows asymptotically faster than* $f$.

Prove or disprove each of the following statements.

(a) $f(m) = 100m^3 + 10m^2 + m$ grows asymptotically slower than $g(m) = 0.001 \cdot m^5$.

**Solution:**

True, since

$$\begin{aligned}
\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{100m^3 + 10m^2 + m}{0.001m^5} \\
&= \lim_{m\to\infty} 10^5 m^{-2} + 10^4 m^{-3} + 10^3 m^{-4} \\
&= 10^5 \lim_{m\to\infty} m^{-2} + 10^4 \lim_{m\to\infty} m^{-3} + 10^3 \lim_{m\to\infty} m^{-4} \\
&= 10^5 \cdot 0 + 10^4 \cdot 0 + 10^3 \cdot 0 = 0.
\end{aligned}$$

(b) $f(m) = \log(m^3)$ grows asymptotically slower than $g(m) = (\log m)^3$.

**Solution:**

True, since

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{\log(n^3)}{(\log n)^3} = \lim_{m\to\infty} \frac{3\log n}{(\log n)^3} = \lim_{m\to\infty} 3 \cdot \frac{1}{(\log n)^2} = 3 \cdot 0 = 0.$$

(c) $f(m) = e^{2m}$ grows asymptotically slower than $g(m) = 2^{3m}$.

***Hint:*** *Recall that for all $n, m \in \mathbb{N}$, we have $n^m = e^{m \ln n}$.*

**Solution:**

True, since

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{e^{2m}}{2^{3m}} = \lim_{m\to\infty} \frac{e^{2m}}{e^{3m\ln 2}} = \lim_{m\to\infty} e^{(2-3\ln 2)m} = \lim_{m\to\infty} e^{(-0.079\ldots)\cdot m} = 0.$$

(d) $f(m) = \sum_{i=1}^{m^2} i$ grows asymptotically slower than $g(m) = \sum_{i=1}^{m} i^2$.

***Hint:*** *You can reuse the inequalities from exercise 1.2.*

**Solution:**

False. With the inequalities from 1.3, we have $\sum_{i=1}^{m^2} i \geq \frac{1}{4}\left(m^2\right)^2 = \frac{1}{4}m^4$ (inequality from 1.3.b with $k = 1$ and $n = m^2$) and $\sum_{i=1}^{m} i^2 \leq m^{2+1} = m^3$ (inequality from 1.3.a with $k = 2$ and $n = m$)[1]. Hence, $\lim_{m\to\infty} \frac{f(m)}{g(m)} \geq \lim_{m\to\infty} \frac{\frac{1}{4}m^4}{m^3} = \lim_{m\to\infty} \frac{1}{4}m = +\infty$, and $f$ does not grow asymptotically slower than $g$.

(e)* If $f(m)$ grows asymptotically slower than $g(m)$, then $\log(f(m))$ grows asymptotically slower than $\log(g(m))$.

**Solution:**

False. Consider $f(m) = m$ and $g(m) = m^2$. We have $\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{m}{m^2} = \lim_{m\to\infty} \frac{1}{m} = 0$, hence $f$ grows asymptotically slower than $g$. However, $\log(f(m)) = \log m$ and $\log(g(m)) = \log\left(m^2\right) = 2\log m$, therefore $\lim_{m\to\infty} \frac{\log(f(m))}{\log(g(m))} = \lim_{m\to\infty} \frac{\log m}{2\log m} = \frac{1}{2} \neq 0$ and $\log(f(m))$ does not grow asymptotically slower than $\log(g(m))$.

(f)* $f(m) = \log(\sqrt{\log(m)})$ grows asymptotically slower than $g(m) = \sqrt{\log(\sqrt{m})}$.

***Hint:*** *You can use L'Hôpital's rule from sheet 0.*

**Solution:**

[1] You can also show this from 1.2

True, since

$$\begin{aligned}
\lim_{m\to\infty}\frac{f(m)}{g(m)} &= \lim_{m\to\infty}\frac{\log(\sqrt{\log(m)})}{\sqrt{\log(\sqrt{m})}} \\
&= \lim_{m\to\infty}\frac{\left(\log\left(\sqrt{\log(m)}\right)\right)'}{\left(\sqrt{\log\left(\sqrt{m}\right)}\right)'} && \text{(L'Hôpital's rule)} \\
&= \lim_{m\to\infty}\frac{\frac{1}{2m\log m}}{\frac{1}{4m\sqrt{\log(\sqrt{m})}}} = \lim_{m\to\infty}\frac{2\sqrt{\log\left(\sqrt{m}\right)}}{\log m} \\
&= \lim_{m\to\infty}\frac{\left(2\sqrt{\log\left(\sqrt{m}\right)}\right)'}{(\log m)'} && \text{(L'Hôpital's rule again)} \\
&= \lim_{m\to\infty}\frac{\frac{1}{m\sqrt{\log(\sqrt{m})}}}{\frac{1}{m}} = \lim_{m\to\infty}\frac{1}{\log\left(\sqrt{m}\right)} = 0.
\end{aligned}$$

**Exercise 1.5** *Proving Inequalities.*

(a) By induction, prove the inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}, \quad n \geq 1.$$

**Solution:**

- **Base Case.**
  For $n = 1$:
  $$\frac{1}{2} \leq \frac{1}{\sqrt{4}},$$
  which is an equality.
- **Induction Hypothesis.**
  Now we assume that it is true for $n = k$, i.e.,
  $$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}}.$$
- **Induction Step.**
  We will prove that it is also true for $n = k + 1$.
  $$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2k-1}{2k}\cdot\frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}}$$
  .

  Plugging in the induction hypothesis, it is sufficient to prove.
  $$\frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{3k+4}} \quad \Leftrightarrow$$

.

$$\frac{2k+1}{2k+2} \leq \frac{\sqrt{3k+1}}{\sqrt{3k+4}}$$

.

Rewriting:

$$\begin{aligned}
&\frac{2k+1}{2k+2} \leq \sqrt{\frac{3k+1}{3k+4}} \\
\Leftrightarrow\ &\left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k+1}{3k+4} \\
\Leftrightarrow\ &(4k^2+4k+1)(3k+4) \leq (4k^2+8k+4)(3k+1) \\
\Leftrightarrow\ &12k^3+28k^2+19k+4 \leq 12k^3+28k^2+20k+4 \\
\Leftrightarrow\ &0 \leq k
\end{aligned}$$

Hence it is true for $n = k+1$.

(b)* Replace $3n+1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

**Solution:**

(b) Sometimes it is easier to prove more than less. This simple approach does not work for the weaker inequality as we are using a weaker (and insufficiently so!) induction hypothesis in each step.